CROSSED PRODUCTS FOR ACTIONS OF CROSSED MODULES ON C*-ALGEBRAS

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ABSTRACT. We decompose the crossed product functor for actions of crossed modules of locally compact groups on C^*-algebras into more elementary constructions: taking crossed products by group actions and fibres in C^*-algebras over topological spaces. For this, we extend the theory of partial crossed products from groups to crossed modules; extend Takesaki–Takai duality to Abelian crossed modules; show that equivalent crossed modules have equivalent categories of actions on C^*-algebras; and show that certain crossed modules are automatically equivalent to Abelian crossed modules.

1. INTRODUCTION

A crossed module of locally compact groups $\mathcal{C}$ consists of two locally compact groups H and G with a continuous group homomorphism $\partial\colon H \to G$ and a continuous conjugation action $c\colon G \to \mathrm{Aut}(H)$ such that

$$\partial(c_g(h)) = g\partial(h)g^{-1}, \qquad c_{\partial(h)}(k) = hkh^{-1}$$

for all $g \in G$, $h, k \in H$. Strict actions of crossed modules on C^*-algebras and crossed products for such actions are defined in [1]. Here we are going to factorise this crossed product functor into more elementary operations, namely, taking crossed products for actions of locally compact *groups* and taking fibres in $\mathrm{C}_0(X)$-C^*-algebras ([15]).

We mostly work with the more flexible notion of action by correspondences introduced in [2]. By [2, Theorem 5.3], such actions are Morita–Rieffel equivalent to ordinary strict actions, that is, actions by automorphisms. This requires, however, to stabilise the C^*-algebras involved; and certain induced actions that we need are easier to describe as actions by correspondences. Here we define the 2-category of crossed module actions by correspondences precisely, making explicit some hints in [2]. Then we translate the definition of this 2-category into the language of Fell bundles; this extends results in [2] from locally compact groups to crossed modules. We define saturated Fell bundles over crossed modules and correspondences between Fell bundles over crossed modules so that they are equivalent to actions by correspondences and transformations between such actions. Correspondences between Fell bundles contain Morita–Rieffel equivalence of Fell bundles and representations of Fell bundles as special cases. The crossed product for a crossed module action by correspondences is defined by a universal property for representations.

Let $\mathcal{C}_i = (G_i, H_i, \partial_i, c_i)$ be crossed modules of locally compact groups. We call a diagram $\mathcal{C}_1 \to \mathcal{C}_2 \to \mathcal{C}_3$ of homomorphisms of crossed modules a (strict) extension if the resulting diagrams $G_1 \to G_2 \to G_3$ and $H_1 \to H_2 \to H_3$ are extensions of locally compact groups in the usual sense. If $\mathcal{C}_2$ acts on a C^*-algebra A, then $\mathcal{C}_1$

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also acts on A by restriction of the $\mathcal{C}_2$-action, and we establish that

$$A \rtimes \mathcal{C}_2 \cong (A \rtimes \mathcal{C}_1) \rtimes \mathcal{C}_3$$

for a certain induced action of $\mathcal{C}_3$ by correspondences on $A \rtimes \mathcal{C}_1$.

If $\mathcal{C}_i$ are ordinary groups G_i viewed as crossed modules, then G_1 is a closed normal subgroup of G_2 with quotient G_2/G_1. It is well-known that $A \rtimes G_1$ carries a Green twisted action of (G_2, G_1) such that $(A \rtimes G_1) \rtimes (G_2, G_1) \cong A \rtimes G_2$. Our theorem says that such a Green twisted action may be turned into an action of G_3 by correspondences with the same crossed product. Our proof, in fact, generalises this idea to cover general extensions of crossed modules. Another special case is [1, Theorem 1], which says that $(A \rtimes H) \rtimes \mathcal{C} \cong A \rtimes G$ if $\mathcal{C} = (G, H, \partial, c)$ is a crossed module and A is a G-C*-algebra.

Moreover, we generalise Takesaki–Takai duality to Abelian crossed modules. We call a crossed module Abelian if the group G is Abelian and the action c is trivial (forcing H to be Abelian). Thus Abelian crossed modules are just continuous homomorphisms $\partial\colon H \to G$ between Abelian locally compact groups. The Pontryagin dual $\hat{\mathcal{C}}$ is the crossed module $\hat{\partial}\colon \hat{G} \to \hat{H}$ given by the transpose of ∂. Our duality theorem says that actions of $\mathcal{C}$ are equivalent to actions of the *arrow groupoid* $\hat{G} \ltimes \hat{H}$ associated to $\mathcal{C}$; this is the transformation groupoid for the action of $\hat{G}$ on $\hat{H}$ where $\hat{g} \in \hat{G}$ acts by right translations by $\hat{\partial}(\hat{g})$.

The duality equivalence we obtain maps an action of $\mathcal{C}$ to its crossed product by G, equipped with the dual action of $\hat{G}$ and a canonical $\mathrm{C}_0(\hat{H})$-C*-algebra structure that comes from the original action of H. The inverse equivalence takes the crossed product by $\hat{G}$ and extends the dual G-action on it to an action of $\mathcal{C}$, using the original $\mathrm{C}_0(\hat{H})$-C*-algebra structure. Thus our duality result merely enriches the usual Takesaki–Takai duality by translating the action of H for a crossed module action into a $\mathrm{C}_0(\hat{H})$-C*-algebra on the crossed product and vice versa.

In this setting, the crossed product for crossed module actions is equivalent to the functor of taking the fibre at $1 \in \hat{H}$ for an action of $\hat{G} \ltimes \hat{H}$. Thus crossed products for Abelian crossed modules may be computed in two steps: first take a crossed product by an action of the Abelian group G, then take a fibre for a $\mathrm{C}_0(\hat{H})$-C*-algebra structure.

Now we describe our factorisation of crossed products for a general crossed module $\mathcal{C} = (G, H, \partial, c)$. Let G_1 be the trivial group and $H_1 := \ker \partial$. Since $\ker \partial$ is Abelian, the trivial maps ∂_1 and c_1 provide a crossed module $\mathcal{C}_1 = (G_1, H_1, \partial_1, c_1)$. This fits into an extension $\mathcal{C}_1 \rightarrowtail \mathcal{C} \twoheadrightarrow \mathcal{C}_2$, $\mathcal{C}_2 = (G_2, H_2, \partial_2, c_2)$, where $G_2 := G$, $H_2 := H/H_1$ and ∂_2 and c_2 are the canonical induced maps.

Next, we let $G_3 := \overline{\partial_2(H_2)} \subseteq G_2$, $H_3 := H_2$ and let ∂_3 and c_3 be the restrictions of ∂_2 and c_2. The crossed module $\mathcal{C}_3 = (G_3, H_3, \partial_3, c_3)$ has the feature that ∂_3 is injective with dense range; we call such crossed modules *thin*.

Since G_3 is a closed normal subgroup of G, $G_4 := G/G_3$ is a locally compact group. Let $H_4 = 0$, ∂_4 and c_4 be trivial. This gives another strict extension of crossed modules $\mathcal{C}_3 \to \mathcal{C}_2 \to \mathcal{C}_4$. Using partial crossed products twice, we get

$$A \rtimes \mathcal{C} \cong (A \rtimes \mathcal{C}_1) \rtimes \mathcal{C}_2 \cong ((A \rtimes \mathcal{C}_1) \rtimes \mathcal{C}_3) \rtimes \mathcal{C}_4.$$

Thus it remains to study crossed products by crossed modules of the special forms $\mathcal{C}_1$, $\mathcal{C}_3$ and $\mathcal{C}_4$, where $G_1 = 0$, $\mathcal{C}_3$ is thin, and $H_4 = 0$.

Since $G_1 = 0$, $\mathcal{C}_1$ is a very particular Abelian crossed module. Here our duality says that actions of $\mathcal{C}_1$ on A are equivalent to $\mathrm{C}_0(\widehat{H_1})$-C*-algebra structures on A, where $\widehat{H_1}$ denotes the dual group of H_1. The crossed product with $\mathcal{C}_1$ is the fibre at $1 \in \widehat{H_1}$ for the corresponding $\mathrm{C}_0(\widehat{H_1})$-C*-algebra structure.

Since $H_4 = 0$, an action of $\mathcal{C}_4$ is equivalent to an action of the group G_4; crossed products also have the usual meaning.

To understand crossed products for the thin crossed module $\mathcal{C}_3$, we replace $\mathcal{C}_3$ by a simpler but equivalent crossed module. Equivalent crossed modules have equivalent categories of actions on C*-algebras, and the equivalence preserves both the underlying C*-algebra of the action and the crossed products. Surprisingly, most thin crossed modules are equivalent to Abelian crossed modules. In fact, we show that all thin crossed modules of Lie groups are equivalent to Abelian crossed modules. In general, not every thin crossed module of locally compact groups is equivalent to an Abelian one (see Example 3.16). However, a sufficient condition for Abelianness of a thin locally compact (not necessarily Lie) crossed module is that H contains a compactly generated subgroup K such that the closed subgroup $\overline{\partial(K)}$ in G is open. This always happens if H itself is compactly generated because $\partial(H)$ is dense in G by assumption.

If the thin crossed module $\mathcal{C}_3$ is equivalent to an Abelian crossed module $\mathcal{C}_5$, then we may turn a $\mathcal{C}_3$-action on A into a $\mathcal{C}_5$-action on A. Both actions have isomorphic crossed products. And by our duality theory, the crossed product is the fibre at $1 \in \widehat{H_5}$ for a canonical $\mathrm{C}_0(\widehat{H_5})$-C*-algebra structure on $A \rtimes G_5$. Furthermore, we may choose $\mathcal{C}_5$ so that G_5 is compact.

Assuming that $\mathcal{C}_3$ satisfies the mild condition to make it Abelian, we thus decompose the crossed product functor for our original crossed module $\mathcal{C}$ into four more elementary steps: taking fibres in $\mathrm{C}_0(X)$-C*-algebras twice and taking crossed products by ordinary groups twice. One of the groups by which we take crossed products may be chosen to be compact and Abelian, the other is $G/\overline{\partial(H)}$.

2. Crossed module actions by correspondences

A *correspondence* between two C^*-algebras A and B is a Hilbert B-module $\mathcal{E}$ with a nondegenerate *-homomorphism $A \to \mathbb{B}(\mathcal{E})$. Correspondences are the arrows of a weak 2-category $\mathfrak{Corr}(2)$, with C^*-algebras as objects and isomorphisms of correspondences as 2-arrows (see [2]).

Let $\mathcal{C} = (H, G, \partial, c)$ be a crossed module of locally compact groups. We may turn $\mathcal{C}$ into a strict 2-group with group of arrows G and 2-arrow space $G \times H$, where (g, h) gives a 2-arrow $g \Rightarrow g\partial(h)$.

Following [2, Section 4], we define an *action of $\mathcal{C}$ by correspondences* as a morphism $\mathcal{C} \to \mathfrak{Corr}(2)$ in the sense of [2, Definition 4.1], with continuity conditions added as in [2, Section 4.1]; we define *transformations* between such actions as in [2, Section 4.2], again with extra continuity requirements; and we define *modifications* between such transformations as in [2, Section 4.3]. This defines a 2-category $\mathfrak{Corr}(\mathcal{C})$.

The definitions in [2] for actions of general strict 2-categories may be simplified because $\mathcal{C}$ is a strict 2-group. Analogous simplifications are already discussed in detail in [2] for weak actions by automorphisms, that is, morphisms to the 2-category $\mathfrak{C}^*(2)$ (see [2] for the definition of $\mathfrak{C}^*(2)$). For this reason, we merely state the simplified definitions without proving that they are equivalent to those in [2, Section 4].

Definition 2.1. Let $\mathcal{C} = (H, G, \partial, c)$ be a crossed module of locally compact groups. An *action of $\mathcal{C}$ by correspondences* consists of

- a C*-algebra A;
- correspondences $\alpha_g \colon A \to A$ for all $g \in G$ with $\alpha_1 = A$ the identity correspondence;

- isomorphisms of correspondences $\omega_{g_1,g_2}\colon \alpha_{g_2}\otimes_A\alpha_{g_1}\to\alpha_{g_1g_2}$ for $g_1,g_2\in G$, where $\omega_{1,g}$ and $\omega_{g,1}$ are the canonical isomorphisms $\alpha_g\otimes_A A\cong\alpha_g$ and $A\otimes_A\alpha_g\cong\alpha_g$;
- isomorphisms of correspondences $\eta_h\colon A\to\alpha_{\partial(h)}$ for all $h\in H$, where $\eta_1=\mathrm{Id}_A$;
- a $\mathrm{C}_0(G)$-linear correspondence α from $\mathrm{C}_0(G,A)$ to itself with fibres α_g; more explicitly, this means a space of continuous sections $\alpha\subseteq\prod_{g\in G}\alpha_g$ such that pointwise products of elements in α with elements of $\mathrm{C}_0(G,A)$ on the left or right are again in α, pointwise inner products of elements in α are in $\mathrm{C}_0(G,A)$, and the projections $\alpha\to\alpha_g$ are all surjective;

these must satisfy the following conditions:

(1) $\omega_{g_1g_2,g_3}(\mathrm{Id}_{\alpha_{g_3}}\otimes_A\omega_{g_1,g_2})=\omega_{g_1,g_2g_3}(\omega_{g_2,g_3}\otimes_A\mathrm{Id}_{\alpha_{g_1}})$ for all $g_1,g_2,g_3\in G$;
(2) $\omega_{\partial(h_1),\partial(h_2)}(\eta_{h_2}\otimes_A\eta_{h_1})=\eta_{h_1h_2}$ for all $h_1,h_2\in H$;
(3) the following diagram of isomorphisms commutes for all $h\in H$, $g\in G$:

$$\begin{tikzcd}
A\otimes_A\alpha_g \arrow[r, "\mathrm{can}"] \arrow[d, "\eta_h\otimes_A\mathrm{Id}_{\alpha_g}"'] & \alpha_g & \alpha_g\otimes_A A \arrow[l, "\mathrm{can}"'] \arrow[d, "\mathrm{Id}_{\alpha_g}\otimes_A\eta_{c_g(h)}"] \\
\alpha_{\partial(h)}\otimes_A\alpha_g \arrow[d, "\omega_{g,\partial(h)}"'] & & \alpha_g\otimes_A\alpha_{\partial(c_gh)} \arrow[d, "\omega_{\partial(c_gh),g}"] \\
\alpha_{g\partial(h)} \arrow[rr, equal] & & \alpha_{\partial(c_gh)g}
\end{tikzcd}\tag{2.2}$$

(4) fibrewise application of ω_{g_2,g_1} gives an isomorphism

$$\omega\colon \pi_2^*\alpha\otimes_{\mathrm{C}_0(G\times G,A)}\pi_1^*\alpha\to\mu^*\alpha,$$

where π_1 and π_2 are the two coordinate projections $G\times G\to G$ and μ is the multiplication map $G\times G\to G$; notice that the fibres of these two Hilbert modules over $\mathrm{C}_0(G\times G,A)$ at $(g_1,g_2)\in G\times G$ are $\alpha_{g_2}\otimes_A\alpha_{g_1}$ and $\alpha_{g_1g_2}$, respectively;
(5) fibrewise application of $(\eta_h)_{h\in H}$ gives an isomorphism $\eta\colon\mathrm{C}_0(H,A)\to\partial^*\alpha$.

Since the maps ω_{g_1,g_2} and η_h are isomorphisms, the continuity conditions (5) and (5) hold if $(\omega_{g_1,g_2})_{g_1,g_2\in G}$ maps $\pi_2^*\alpha\otimes_A\pi_1^*\alpha\subseteq\prod_{g_1,g_2\in G}\alpha_{g_2}\otimes_A\alpha_{g_1}$ into $\mu^*\alpha\subseteq\prod_{g_1,g_2\in G}\alpha_{g_1g_2}$ and $(\eta_h)_{h\in H}$ maps $\mathrm{C}_0(H,A)\subseteq\prod_{h\in H}A$ into $\partial^*\alpha\subseteq\prod_{h\in H}\alpha_{\partial(h)}$; these maps are automatically unitary (isometric and surjective).

The *trivial* $\mathcal{C}$-action on B is given by $\beta_g=B$ for $g\in G$, $\omega_{g_2,g_1}=\mathrm{Id}_B$ for $g_1,g_2\in G$, $\eta_h=\mathrm{Id}_B$ for $h\in H$, and $\beta=\mathrm{C}_0(G,B)$.

Definition 2.3. Let $(A,\alpha,\omega^A,\eta^A)$ and $(B,\beta,\omega^B,\eta^B)$ be actions of $\mathcal{C}$ by correspondences. A *$\mathcal{C}$-equivariant correspondence or transformation* from A to B consists of

- a correspondence $\mathcal{E}$ from A to B, and
- isomorphisms of correspondences $\chi_g\colon\mathcal{E}\otimes_B\beta_g\to\alpha_g\otimes_A\mathcal{E}$ with $\chi_1=1$,

such that

(1) for all $h\in H$, the following diagram commutes:

$$\begin{tikzcd}
A\otimes_A\mathcal{E} \arrow[r, "\mathrm{can}"] \arrow[d, "\eta_h^A\otimes_A\mathrm{Id}_{\mathcal{E}}"'] & \mathcal{E} & \mathcal{E}\otimes_B B \arrow[l, "\mathrm{can}"'] \arrow[d, "\mathrm{Id}_{\mathcal{E}}\otimes_B\eta_h^B"] \\
\alpha_{\partial(h)}\otimes_A\mathcal{E} & & \mathcal{E}\otimes_B\beta_{\partial(h)} \arrow[ll, "\chi_{\partial(h)}"]
\end{tikzcd}\tag{2.4}$$

(2) for all $g_1, g_2 \in G$, the following diagram commutes:

$$\begin{tikzcd} \mathcal{E} \otimes_B \beta_{g_2} \otimes_B \beta_{g_1} \arrow[rr, "\chi_{g_2} \otimes_B \mathrm{Id}_{\beta_{g_1}}"] \arrow[d, "\mathrm{Id}_{\mathcal{E}} \otimes_B \omega^B_{g_1,g_2}"'] & & \alpha_{g_2} \otimes_A \mathcal{E} \otimes_B \beta_{g_1} \arrow[d, "\mathrm{Id}_{\alpha_{g_2}} \otimes_A \chi_{g_1}"] \\ \mathcal{E} \otimes_B \beta_{g_1 g_2} \arrow[dr, "\chi_{g_1 g_2}"'] & & \alpha_{g_2} \otimes_A \alpha_{g_1} \otimes_A \mathcal{E} \arrow[dl, "\omega^A_{g_1,g_2} \otimes_A \mathrm{Id}_{\mathcal{E}}"] \\ & \alpha_{g_1 g_2} \otimes_A \mathcal{E} & \end{tikzcd} \tag{2.5}$$

(3) pointwise application of χ_g gives a $\mathrm{C}_0(G)$-linear isomorphism $\chi\colon \mathcal{E} \otimes_B \beta \to \alpha \otimes_A \mathcal{E}$.

A transformation is called an (equivariant Morita–Rieffel) *equivalence* if $\mathcal{E}$ is an equivalence, that is, the left A-action is given by an isomorphism $A \cong \mathbb{K}(\mathcal{E})$.

A transformation is called a (covariant) *representation* of (A, α) on B if the $\mathcal{C}$-action on B is trivial and $\mathcal{E}$ is the correspondence associated to a nondegenerate *-homomorphism $\pi\colon A \to \mathcal{M}(B)$. In this case, we also write $\pi\colon (A, \alpha) \to B$ or simply $\pi\colon A \to B$ to denote the representation.

The continuity condition (3) is equivalent to $(\chi_g)_{g\in G}$ mapping $\mathcal{E} \otimes_B \beta \subseteq \prod_{g\in G} \mathcal{E} \otimes_B \beta_g$ into $\alpha \otimes_A \mathcal{E} \subseteq \prod_{g\in G} \alpha_g \otimes_A \mathcal{E}$; this map is automatically unitary because it is fibrewise unitary.

Definition 2.6. A *modification* between two transformations $(\mathcal{E}, \chi_g)$ and $(\mathcal{E}', \chi'_g)$ from A to B is a unitary $W\colon \mathcal{E} \to \mathcal{E}'$ such that for all $g \in G$ the following diagram commutes:

$$\begin{tikzcd} \mathcal{E} \otimes_B \beta_g \arrow[r, "\chi_g"] \arrow[d, "W \otimes_B \mathrm{Id}_{\beta_g}"'] & \alpha_g \otimes_A \mathcal{E} \arrow[d, "\mathrm{Id}_{\alpha_g} \otimes_A W"] \\ \mathcal{E}' \otimes_B \beta_g \arrow[r, "\chi'_g"'] & \alpha_g \otimes_A \mathcal{E}' \end{tikzcd} \tag{2.7}$$

The notion of representation above leads to a definition of crossed products:

Definition 2.8. A *crossed product* for a $\mathcal{C}$-action $(A, \alpha, \omega, \eta)$ by correspondences is a C^*-algebra B with a representation $\pi^{\mathrm{u}}\colon A \to B$ that is universal in the sense that any other representation $A \to C$ factors uniquely as $f \circ \pi^{\mathrm{u}}$ for a morphism (nondegenerate *-homomorphism) $f\colon B \to \mathcal{M}(C)$.

If a crossed product exists, then its universal property determines it uniquely up to canonical isomorphism because an isomorphism in the morphism category of C*-algebras must be a *-isomorphism in the usual sense. We will construct crossed products later using cross-sectional C^*-algebras of Fell bundles.

The actions, transformations, and modifications defined above are the objects, arrows and 2-arrows of a weak 2-category (that is, bicategory) $\mathfrak{Corr}(\mathcal{C})$ with invertible 2-arrows. This statement contains the following assertions. Given two $\mathcal{C}$-actions $x_i = (A_i, \alpha_i, \omega_i, \eta_i)$ for $i = 1, 2$, the transformations $x_1 \to x_2$ (as objects) and the modifications (as morphisms) between such transformations form a groupoid $\mathfrak{Corr}_{\mathcal{C}}(x_1, x_2)$; here the composition of modifications is just the composition of unitary operators. Given three $\mathcal{C}$-actions x_1, x_2 and x_3, there is a composition bifunctor

$$\mathfrak{Corr}_{\mathcal{C}}(x_2, x_3) \times \mathfrak{Corr}_{\mathcal{C}}(x_1, x_2) \to \mathfrak{Corr}_{\mathcal{C}}(x_1, x_3);$$

the composition of two transformations $(\mathcal{E}_1, \chi_1)$ from x_1 to x_2 and $(\mathcal{E}_2, \chi_2)$ from x_2 to x_3 is the transformation from x_1 to x_3 consisting of $\mathcal{E}_1 \otimes_{A_2} \mathcal{E}_2$ and

$$\mathcal{E}_1 \otimes_{A_2} \mathcal{E}_2 \otimes_{A_3} \alpha_{3g} \xrightarrow{\mathrm{Id}_{\mathcal{E}_1} \otimes_{A_2} \chi_{2g}} \mathcal{E}_1 \otimes_{A_2} \alpha_{2g} \otimes_{A_1} \mathcal{E}_2 \xrightarrow{\chi_{1g} \otimes_{A_1} \mathrm{Id}_{\mathcal{E}_2}} \alpha_{1g} \otimes_{A_1} \mathcal{E}_1 \otimes_{A_2} \mathcal{E}_2.$$

This composition is associative and unital up to canonical isomorphisms with suitable coherence properties. To see the associators, notice that $(\mathcal{E}_1 \otimes_{A_2} \mathcal{E}_2) \otimes_{A_3} \mathcal{E}_3$ and $\mathcal{E}_1 \otimes_{A_2} (\mathcal{E}_2 \otimes_{A_3} \mathcal{E}_3)$ are not identical but merely canonically isomorphic. These canonical isomorphisms are the associators. They are canonical enough that being careful about them would lead to more confusion than leaving them out. The identity arrow on A is A with the canonical isomorphisms $\chi_g\colon A \otimes_A \alpha_g \cong \alpha_g \cong \alpha_g \otimes_A A$. The composition of another arrow with such an identity arrow is canonically isomorphic to that arrow using the canonical isomorphisms $A \otimes_A \mathcal{E} \cong \mathcal{E}$ and $\mathcal{E} \otimes_B B \cong \mathcal{E}$. These are the unit transformations. The coherence conditions for a weak 2-category listed in [2, Section 2.2.1] are trivially satisfied.

2.1. **Translation to Fell bundles.** For a locally compact group, it is shown in [2] that actions by correspondences are equivalent to saturated Fell bundles. When we reinterpret everything in terms of Fell bundles, transformations become correspondences between Fell bundles, modifications become isomorphisms of such correspondences, and representations become representations of Fell bundles in the usual sense. Thus the cross-sectional C^*-algebra of the associated Fell bundle has the correct universal property for a crossed product. We want to extend all these results to crossed module actions.

Most of the work is already done in [2]. Let $\mathcal{C} = (G, H, \partial, c)$ be a crossed module of locally compact groups and let $(A, \alpha, \omega, \eta)$ be an action of $\mathcal{C}$ as in Definition 2.1. Forgetting η, the data (A, α, ω) is a continuous action by correspondences of the locally compact group G (as defined in [2] or as in our Definition 2.1 for G viewed as a crossed module). Results in [2] about locally compact groups show that the data (A, α, ω) is equivalent to a saturated Fell bundle over G. The map η gives us some extra data that describes how the group H acts. A transformation between actions of $\mathcal{C}$ is the same as a transformation between the resulting actions of G that satisfies an additional compatibility condition with the H-actions. And modifications for G- and $\mathcal{C}$-actions are just the same.

It remains to translate everything in $\mathfrak{Corr}(\mathcal{C})$ related to the group H to the language of Fell bundles. For this we first need some notation.

Let $\mathcal{A}$ be a Fell bundle over G with fibres $\mathcal{A}_g$ at $g \in G$. A *multiplier of order* $g \in G$ of $\mathcal{A}$ is a pair $\mu = (L, R)$ (left and right multipliers) of maps $L, R\colon \mathcal{A} \to \mathcal{A}$ such that $L(\mathcal{A}_f) \subseteq \mathcal{A}_{gf}$ and $R(\mathcal{A}_f) \subseteq \mathcal{A}_{fg}$ for all $f \in G$, and $aL(b) = R(a)b$ for all $a, b \in \mathcal{A}$ (see [3, VIII.2.14]). We write $\mathcal{M}(\mathcal{A})_g$ for the set of multipliers of order g. We usually write $\mu \cdot a = L(a)$ and $a \cdot \mu = R(a)$.

The maps L and R must be fibrewise linear and bounded. The adjoint of μ is defined by $\mu^* \cdot a = (a^* \cdot \mu)^*$ and $a \cdot \mu^* = (\mu \cdot a^*)^*$, and μ is called unitary if $\mu^*\mu = \mu\mu^* = 1$ (the unit of $\mathcal{M}(\mathcal{A}_1)$). The set $\mathcal{M}(\mathcal{A}) = \bigcup_{g\in G} \mathcal{M}(\mathcal{A})_g$ of all multipliers of $\mathcal{A}$ is a Fell bundle over G viewed as a discrete group, called the *multiplier Fell bundle* of $\mathcal{A}$. We endow $\mathcal{M}(\mathcal{A})$ with the *strict topology*: a net (μ_i) in $\mathcal{M}(\mathcal{A})$ converges strictly to $\mu \in \mathcal{M}(\mathcal{A})$ if and only if $\mu_i \cdot a \to \mu \cdot a$ and $a \cdot \mu_i \to a \cdot \mu$ in $\mathcal{A}$ for all $a \in \mathcal{A}$. Let $\mathcal{UM}(\mathcal{A})$ be the group of unitary multipliers of $\mathcal{A}$ of arbitrary order.

We may view the fibres $\mathcal{A}_g$ as Hilbert bimodules over $\mathcal{A}_1$ using the multiplication in the Fell bundle and the inner products $\langle x, y\rangle_{\mathcal{A}_1} = x^*y$ on the right and ${}_{\mathcal{A}_1}\langle x, y\rangle = xy^*$ on the left. Taking these operations fibrewise makes the space $\Gamma_0(\mathcal{A})$ of continuous sections of $\mathcal{A}$ vanishing at infinity a Hilbert bimodule over $\mathrm{C}_0(G, \mathcal{A}_1)$ because the multiplication and involution in the Fell bundle are continuous.

Lemma 2.9. *Let$\mathcal{A}$ be a saturated Fell bundle. Then $\mathcal{M}(\mathcal{A})_g$ is isomorphic to the space of adjointable operators $\mathcal{A}_1 \to \mathcal{A}_g$. The space of strictly continuous sections of $\mathcal{M}(\mathcal{A})$ is isomorphic to the space of adjointable operators $\mathrm{C}_0(G, \mathcal{A}_1) \to \Gamma_0(\mathcal{A})$, that is, the multiplier Hilbert bimodule of $\Gamma_0(\mathcal{A})$ (as defined in* [6, Chapter 1.2]*).*

Proof. A multiplier μ of $\mathcal{A}$ of order g restricts to an adjointable map $\mathcal{A}_1 \to \mathcal{A}_g$, $a \mapsto \mu \cdot a$, with adjoint $b \mapsto \mu^* \cdot b$. Furthermore, since $\mathcal{A}_1 \cdot \mathcal{A}_{g_2} = \mathcal{A}_{g_2}$ for all $g_2 \in G$, an adjointable map $\mathcal{A}_1 \to \mathcal{A}_g$ extends uniquely to a multiplier of $\mathcal{A}$. For the last statement, we must show that a section $(\mu_g)_{g\in G}$ of $\mathcal{M}(\mathcal{A})$ is strictly continuous if and only if pointwise application of μ_g and μ_g^* gives well-defined maps $\mathrm{C}_0(G,\mathcal{A}_1) \leftrightarrow \Gamma_0(\mathcal{A})$. The existence of a map $\mathrm{C}_0(G,\mathcal{A}_1) \to \Gamma_0(\mathcal{A})$ is equivalent to the continuity of $\mu_g \cdot a$ for all $a \in \mathcal{A}_1$, which is equivalent to the continuity of $\mu_g \cdot a$ for all $a \in \mathcal{A}$ because $\mathcal{A} = \mathcal{A}_1 \cdot \mathcal{A}$. Since the pointwise product maps for a saturated Fell bundle satisfy

$$\mathrm{C}_0(G,\mathcal{A}_1) \cdot \Gamma_0(\mathcal{A}) = \Gamma_0(\mathcal{A}) \quad \text{and} \quad \Gamma_0(\mathcal{A}) \cdot \Gamma_0(\mathcal{A}^*) = \mathrm{C}_0(G,\mathcal{A}_1),$$

(with $\mathcal{A}_g^* := \mathcal{A}_{g^{-1}}$), the continuity of $\mu_g^* \cdot a_g$ for all $(a_g) \in \Gamma_0(\mathcal{A})$ is equivalent to the continuity of $\mu_g^* \cdot a_g$ for all $(a_g) \in \mathrm{C}_0(G,\mathcal{A}_1)$. This is in turn equivalent to the continuity of $a \cdot \mu_g$ for all $a \in \mathcal{A}_1$ or to the continuity of $a \cdot \mu_g$ for all $a \in \mathcal{A}$. ☐

Definition 2.10. Let $\mathcal{C} = (G, H, \partial, c)$ be a crossed module of locally compact groups. A *Fell bundle* over $\mathcal{C}$ is a Fell bundle $\mathcal{A} = (\mathcal{A}_g)_{g\in G}$ over G together with a strictly continuous group homomorphism $\upsilon\colon H \to \mathcal{UM}(\mathcal{A})$, such that

(1) $\upsilon_h \in \mathcal{M}(\mathcal{A})_{\partial(h)}$ for all $h \in H$, and
(2) $a \cdot \upsilon_h = \upsilon_{c_g(h)} \cdot a$ for all $a \in \mathcal{A}_g$ and $h \in H$.

A Fell bundle over $\mathcal{C}$ is *saturated* if it is saturated as a Fell bundle over G.

Theorem 2.11. *Actions of $\mathcal{C}$ on C^*-algebras by correspondences are equivalent to saturated Fell bundles over $\mathcal{C}$.*

Proof. Let $(A, \alpha, \omega, \eta)$ be an action of $\mathcal{C}$ by correspondences as in Definition 2.1.

The data A, α, ω subject to the conditions (1) and (4) in Definition 2.1 are an action of the locally compact group G by correspondences. We have simplified this compared to the definition in [2] by requiring the unit transformation $A \to \alpha_1$ to be the identity. It is shown as in the proof of [2, Lemma 3.7] that any action of G by correspondences is equivalent to one with this extra property. Thus our data (A, α, ω) is equivalent to a saturated Fell bundle over G by [2, Theorem 3.17]. Its fibres are $\mathcal{A}_g = \alpha_{g^{-1}}$; the multiplication $\mathcal{A}_{g_1} \times \mathcal{A}_{g_2} \to \mathcal{A}_{g_1g_2}$ is $a \cdot b := \omega_{g_2^{-1}, g_1^{-1}}(a \otimes b)$; the involution is the unique one for which the inner product on $\mathcal{A}_g = \alpha_{g^{-1}}$ has the expected form: $\langle x, y\rangle = x^* \cdot y$.

An isomorphism of right Hilbert A-modules $\eta_{h^{-1}}\colon A \to \alpha_{\partial(h^{-1})} = \mathcal{A}_{\partial(h)}$ is equivalent to a unitary multiplier υ_h of $\mathcal{A}$ of order $\partial(h)$ by Lemma 2.9. We claim that the conditions in Definition 2.10 for this map $\upsilon\colon H \to \mathcal{UM}(\mathcal{A})$ are equivalent to the conditions (2), (3) and (5) in Definition 2.1.

The condition $\omega_{\partial(h_1),\partial(h_2)}(\eta_{h_2} \otimes_A \eta_{h_1}) = \eta_{h_1h_2}$ is equivalent to υ being a group homomorphism. The diagram (2.2) commutes if and only if $\upsilon_h^* \cdot a_1 \cdot a_g \cdot a_1' = a_1 \cdot a_g \cdot \upsilon_{c_g(h)}^* \cdot a_1'$ for all $a_1, a_1' \in \mathcal{A}_1$, $a_g \in \mathcal{A}_g$, $g \in G$, $h \in H$; here we use $\mathcal{A}_1 \cdot \mathcal{A}_g \cdot \mathcal{A}_1 = \mathcal{A}_g$ and that $\upsilon_h^* = \upsilon_{h^{-1}}$. Letting a_1 and a_1' run through approximate units, we get $\upsilon_h^* \cdot a_g = a_g \cdot \upsilon_{c_g(h)}^*$ for all $h \in H$, $g \in G$, $a_g \in \mathcal{A}_g$, which is equivalent to condition (2) in Definition 2.10.

The equivalence of the continuity conditions in Definitions 2.1 and 2.10 follows as in the proof of the second statement in Lemma 2.9. ☐

Since actions by automorphisms may be viewed as actions by correspondences, Theorem 2.11 implies that weak actions by automorphisms as considered in [2, Section 4.1.1] give rise to saturated Fell bundles. We make this more explicit for strict actions:

Example 2.12. Let (α, u) be an action of $\mathcal{C} = (G, H, \partial, c)$ by *-automorphisms on a C^*-algebra A (as defined in [1, Definition 3.1]). That is, $\alpha\colon G \to \mathrm{Aut}(A)$ is a

(strongly continuous) action of G on A by *-automorphisms and $u\colon H \to \mathcal{UM}(A)$ is a strictly continuous group homomorphism, and

(1) $\alpha_{\partial(h)}(a) = u_h a u_h^*$ for all $a \in A$ and $h \in H$; and
(2) $\alpha_g(u_h) = u_{c_g(h)}$ for all $g \in G$ and $h \in H$.

Let $\mathcal{A} = A \times_\alpha G$ be the semidirect product Fell bundle over G for the action α (see [3, VIII.4]). Its operations are given by

$$(a,f)\cdot(b,g) = (a\alpha_f(b), fg) \quad\text{and}\quad (a,g)^* = (\alpha_{g^{-1}}(a^*), g^{-1})$$

for all $a,b \in A$ and $f,g \in G$. The multiplier Fell bundle $\mathcal{M}(\mathcal{A})$ is isomorphic to the semidirect product Fell bundle $\mathcal{M}(\mathcal{A}) \times_\alpha G$, where α is tacitly extended to a G-action on $\mathcal{M}(\mathcal{A})$. Therefore, the formulas

$$\upsilon_h \cdot (a,g) := (au_h^*, \partial(h)g) \quad\text{and}\quad (a,g)\cdot \upsilon_h := (au^*_{c_g(h)}, g\partial(h))$$

for $h \in H$, $g \in G$, $a \in A$ define a unitary multiplier υ_h of order $\partial(h)$ of $\mathcal{A}$, where $\upsilon_h = (u_h^*, \partial(h))$. The pair $(\mathcal{A}, \upsilon)$ is a (saturated) Fell bundle over $\mathcal{C}$ in the above sense.

Next we translate transformations between $\mathcal{C}$-actions into correspondences between Fell bundles over $\mathcal{C}$. Let $(\mathcal{A}, \upsilon^{\mathcal{A}})$ and $(\mathcal{B}, \upsilon^{\mathcal{B}})$ be Fell bundles over $\mathcal{C}$. The following definition is an extension of [2, Definition 3.21] from groups to 2-groups (that is, crossed modules). It should also be compared with the notion of equivalence between Fell bundles over groupoids appearing in [12, 14, 17].

Definition 2.13. A *$\mathcal{C}$-equivariant correspondence* from $(\mathcal{A}, \upsilon^{\mathcal{A}})$ and $(\mathcal{B}, \upsilon^{\mathcal{B}})$ is a continuous Banach bundle $\mathcal{E} = (\mathcal{E}_g)_{g\in G}$ over G together with

- a continuous multiplication $\mathcal{A} \times \mathcal{E} \to \mathcal{E}$ that maps $\mathcal{A}_{g_1} \times \mathcal{E}_{g_2}$ to $\mathcal{E}_{g_1 g_2}$;
- a continuous multiplication $\mathcal{E} \times \mathcal{B} \to \mathcal{E}$ that maps $\mathcal{E}_{g_1} \times \mathcal{B}_{g_2}$ to $\mathcal{E}_{g_1 g_2}$;
- a continuous inner product $\langle \sqcup, \sqcup \rangle\colon \mathcal{E} \times \mathcal{E} \to \mathcal{B}$ that maps $\mathcal{E}_{g_1} \times \mathcal{E}_{g_2}$ to $\mathcal{B}_{g_1^{-1} g_2}$;

these must satisfy

(1) associativity $x\cdot(y\cdot z) = (x\cdot y)\cdot z$ for (x,y,z) in $\mathcal{A}\times\mathcal{A}\times\mathcal{E}$, $\mathcal{E}\times\mathcal{B}\times\mathcal{B}$, and $\mathcal{A}\times\mathcal{E}\times\mathcal{B}$;
(2) $\mathcal{A}_1 \cdot \mathcal{E}_g = \mathcal{E}_g = \mathcal{E}_g \cdot \mathcal{B}_1$ for all $g \in G$;
(3) $\xi_2 \mapsto \langle \xi_1, \xi_2\rangle$ is fibrewise linear for all $\xi_1 \in \mathcal{E}$ and $\langle \xi_1, \xi_2\rangle^* = \langle \xi_2, \xi_1\rangle$ for all $\xi_1, \xi_2 \in \mathcal{E}$;
(4) $\langle \xi_1, \xi_2 \cdot b_2\rangle = \langle \xi_1, \xi_2\rangle b_2$ for all $b_2 \in \mathcal{B}$, $\xi_1, \xi_2 \in \mathcal{E}$;
(5) $\langle \xi, \xi\rangle \ge 0$ in $\mathcal{B}_1$ for all $\xi \in \mathcal{E}$, and $\|\xi\|^2 = \|\langle \xi, \xi\rangle\|$;
(6) $\langle a\xi_1, \xi_2\rangle = \langle \xi_1, a^*\xi_2\rangle$ for all $a \in \mathcal{A}$, $\xi_1, \xi_2 \in \mathcal{E}$;
(7) $\upsilon^{\mathcal{A}}_{c_g(h)} \cdot \xi = \xi \cdot \upsilon^{\mathcal{B}}_h$ for $h \in H$, $g \in G$, $\xi \in \mathcal{E}_g$.

An *isomorphism of correspondences* is a homeomorphism $\mathcal{E} \to \mathcal{E}'$ that is compatible with the left and right multiplication maps and the inner product.

Theorem 2.14. *Let $(\mathcal{A}, \upsilon^{\mathcal{A}})$ and $(\mathcal{B}, \upsilon^{\mathcal{B}})$ be saturated Fell bundles over $\mathcal{C}$. The groupoid of Fell bundle correspondences from $(\mathcal{A}, \upsilon^{\mathcal{A}})$ to $(\mathcal{B}, \upsilon^{\mathcal{B}})$ and isomorphisms between such correspondences is equivalent to the groupoid of transformations and modifications between the $\mathcal{C}$-actions associated to $(\mathcal{A}, \upsilon^{\mathcal{A}})$ and $(\mathcal{B}, \upsilon^{\mathcal{B}})$.*

Proof. Let $(\mathcal{E}_1, \chi_g)$ be a transformation from the $\mathcal{C}$-action corresponding to $\mathcal{A}$ to the $\mathcal{C}$-action corresponding to $\mathcal{B}$. This contains isomorphisms

$$\mathcal{E}_g := \mathcal{A}_g \otimes_A \mathcal{E}_1 \xrightarrow[\cong]{\chi_{g^{-1}}} \mathcal{E}_1 \otimes_B \mathcal{B}_g$$

for $g \in G$. The notation $\mathcal{E}_g$ leads to no serious confusion for $g = 1$ because $\mathcal{A}_1 \otimes_A \mathcal{E}_1 = A \otimes_A \mathcal{E}_1 \cong \mathcal{E}_1$ canonically. The spaces $\mathcal{E}_g$ are the fibres of a correspondence

$\Gamma_0(\mathcal{A}) \otimes_A \mathcal{E}_1$ from $\mathrm{C}_0(G, A)$ to $\mathrm{C}_0(G, B)$; the continuity of χ gives $\Gamma_0(\mathcal{A}) \otimes_A \mathcal{E}_1 \cong \mathcal{E}_1 \otimes_B \Gamma_0(\mathcal{B})$. We may topologise $\mathcal{E} := \bigcup_{g\in G} \mathcal{E}_g$ so that

$$\Gamma_0(\mathcal{E}) = \Gamma_0(\mathcal{A}) \otimes_A \mathcal{E}_1 \cong \mathcal{E}_1 \otimes_B \Gamma_0(\mathcal{B}).$$

The multiplications in $\mathcal{A}$ and $\mathcal{B}$ induce continuous multiplication maps $\mathcal{A}\times\mathcal{E} \to \mathcal{E}$ $\mathcal{E}\times\mathcal{B} \to \mathcal{E}$, defining $\mathcal{E}_g$ as $\mathcal{A}_g \otimes_A \mathcal{E}_1$ and $\mathcal{E}_1 \otimes_B \mathcal{B}_g$, respectively. These multiplications satisfy $(a_1a_2)\xi = a_1(a_2\xi)$ for all $a_1, a_2 \in \mathcal{A}$, $\xi \in \mathcal{E}$ and $\xi(b_1b_2)\xi = (\xi b_1)b_2$ for all $\xi \in \mathcal{E}$, $b_1, b_2 \in \mathcal{B}$. The bimodule property $(a\xi)b = a(\xi b)$ for all $a \in \mathcal{A}$, $\xi \in \mathcal{E}$, $b \in \mathcal{B}$ is equivalent to (2.5) by a routine computation. The nondegeneracy of correspondences implies $\mathcal{E}_g = \mathcal{A}_1 \cdot \mathcal{E}_g = \mathcal{E}_g \cdot \mathcal{B}_1$.

The B-valued inner product on $\mathcal{E}$ induces inner product maps

$$\langle \sqcup, \sqcup \rangle \colon \overline{\mathcal{E}}_{g_1} \times \mathcal{E}_{g_2} \to \mathcal{B}_{g_1^{-1} g_2},$$

where $\overline{\mathcal{E}}_{g_1}$ denotes the conjugate space of $\mathcal{E}_{g_1}$ and $\langle \xi_1 \otimes b_1, \xi_2 \otimes b_2\rangle := b_1^* \langle \xi_1, \xi_2\rangle_B b_2$ for $\xi_1, \xi_2 \in \mathcal{E}$, $b_1 \in \mathcal{B}_{g_1}$, $b_2 \in \mathcal{B}_{g_2}$, and we identify $B = \mathcal{B}_1$. The required Hilbert module properties of this inner product are routine to check.

A unitary multiplier u of $\mathcal{A}$ of degree $g \in G$ gives an adjointable map $\mathcal{A}_{g_2} \to \mathcal{A}_{gg_2}$, which induces an adjointable map $\mathcal{E}_{g_2} \to \mathcal{E}_{gg_2}$, $\xi \mapsto u \cdot \xi$. Similarly, a unitary multiplier v of $\mathcal{B}$ of degree $g \in G$ induces maps $\mathcal{E}_{g_2} \to \mathcal{E}_{g_2 g}$, $\xi \mapsto \xi \cdot v$, using the map $\mathcal{B}_{g_2} \to \mathcal{B}_{g_2 g}$. In particular, $\upsilon_h^{\mathcal{A}}$ and $\upsilon_h^{\mathcal{B}}$ act on $(\mathcal{E}_g)$ by left and right multiplication maps. The condition (2.4) for a transformation is equivalent to $\upsilon_h^{\mathcal{A}} \cdot \xi = \xi \cdot \upsilon_h^{\mathcal{B}}$ for all $h \in H$, $\xi \in \mathcal{E}$. For $\xi \in \mathcal{E}_g$ and $g \in G$, this implies

$$\upsilon^{\mathcal{A}}_{c_g(h)} \cdot \xi = \xi \cdot \upsilon^{\mathcal{B}}_h.$$

To see this, write $\xi = a\xi_1$ with $a \in \mathcal{A}_g$, $\xi_1 \in \mathcal{E}_1$, and compute $a\xi_1 \cdot \upsilon_h^B = a\upsilon_h^{\mathcal{A}}\xi_1 = \upsilon^{\mathcal{A}}_{c_g(h)} \cdot a\xi_1$ using condition (2) in Definition 2.10. Thus a transformation between actions of $\mathcal{C}$ by correspondences yields a correspondence between the associated Fell bundles.

To show the converse, take a correspondence of Fell bundles $\mathcal{E}$. Then $\mathcal{E}_g$ is a Hilbert module over $\mathcal{B}_1$, and the multiplication maps $\mathcal{E}_1 \otimes_{\mathcal{B}_1} \mathcal{B}_g \to \mathcal{E}_g$ must be unitary. These maps are the fibres of a continuous map, so that $\mathcal{E} \cong \mathcal{E}_1 \otimes_{\mathcal{B}_1} \mathcal{B}$. The left multiplication gives unitaries

$$\mathcal{A}_g \otimes_{\mathcal{A}_1} \mathcal{E}_1 \cong \mathcal{E}_g \cong \mathcal{E}_1 \otimes_{\mathcal{B}_1} \mathcal{B}_g.$$

These yield the maps $\chi_{g^{-1}} \colon \mathcal{A}_g \otimes_{\mathcal{A}_1} \mathcal{E}_1 \xrightarrow{\sim} \mathcal{E}_1 \otimes_{\mathcal{B}_1} \mathcal{B}_g$. Reversing the computations above, it is then routine to show that $(\mathcal{E}_1, \chi_g)$ is a transformation of $\mathcal{C}$-actions.

A modification between transformations consists of an isomorphism of correspondences $\mathcal{E} \to \mathcal{E}'$, which then induces isomorphisms $\mathcal{E}_g = \mathcal{E} \otimes_B \mathcal{B}_g \cong \mathcal{E}' \otimes_B \mathcal{B}_g = \mathcal{E}'_g$. The compatibility with χ_g shows that these maps preserve also the left module structure, hence give an isomorphism of Fell bundle correspondences. Conversely, an isomorphism of Fell bundle correspondences is determined by its action on the unit fibre $\mathcal{E}_1$, which is a unitary W that satisfies the compatibility condition (2.7).

Thus our constructions of Fell bundle correspondences from $\mathcal{C}$-actions and vice versa are fully faithful functors between the respective categories. These functors are inverse to each other up to natural isomorphisms by construction. ☐

The invertible transformations between $\mathcal{C}$-actions are those where $\mathcal{E}$ is an imprimitivity bimodule. In this case, $\mathcal{E}$ carries an $\mathcal{A}$-valued left inner product as well such that ${}_{\mathcal{A}}\langle \xi, \eta\rangle \cdot \zeta = \xi \cdot \langle \eta, \zeta\rangle_{\mathcal{B}}$ for all $\xi, \eta, \zeta \in \mathcal{E}$. This leads to an appropriate definition of a *Morita–Rieffel equivalence* between Fell bundles over $\mathcal{C}$. The functoriality of crossed products implies that Morita–Rieffel equivalent Fell bundles over $\mathcal{C}$ have Morita–Rieffel equivalent crossed products.

Now we specialise to a transformation $(\mathcal{E}_1, \chi)$ from A to B where the underlying right Hilbert module of $\mathcal{E}_1$ is B itself. Thus the left A-action on $\mathcal{E}_1$ is a nondegenerate *-homomorphism $A \to \mathcal{M}(B)$.

Proposition 2.15. *Transformations $(\mathcal{E}_1, \chi)$ between two $\mathcal{C}$-algebras A and B with $\mathcal{E}_1 = B$ as a Hilbert B-module correspond to morphisms between the Fell bundles $\mathcal{A}$ and $\mathcal{B}$ over $\mathcal{C}$ associated to A and B, that is, to maps $f\colon \mathcal{A} \to \mathcal{M}(\mathcal{B})$*

- *that are fibrewise linear;*
- *satisfy $f(a^*) = f(a)^*$ and $f(a_1 \cdot a_2) = f(a_1) \cdot f(a_2)$ for all $a, a_1, a_2 \in \mathcal{A}$;*
- *that are nondegenerate ($f(\mathcal{A}_1) \cdot \mathcal{B}_1 = \mathcal{B}_1$);*
- *satisfy $f(\upsilon_h^{\mathcal{A}}) = \upsilon_h^{\mathcal{B}}$ for all $h \in H$, where nondegeneracy has been used to extend f to multipliers; and*
- *that are strictly continuous in the sense that the map $\mathcal{A} \times \mathcal{B} \to \mathcal{B}$, $(a, b) \mapsto f(a) \cdot b$, is continuous.*

The joint continuity in the last condition is equivalent to the continuity of the maps $a \mapsto f(a) \cdot b$ for all $b \in \mathcal{B}_1$.

Proof. Let $\mathcal{E}$ be the Fell bundle correspondence associated to a transformation $(\mathcal{E}_1, \chi)$. We assume that $\mathcal{E}_1 = \mathcal{B}_1 = B$ as a right Hilbert $\mathcal{B}_1$-module. This gives an isomorphism of Banach bundles $\mathcal{E} \cong \mathcal{B}$ that is right $\mathcal{B}$-linear and unitary for the $\mathcal{B}$-valued inner product because the multiplication map $(\xi, b) \mapsto \xi \cdot b$ induces a unitary $\mathcal{E}_1 \otimes_{\mathcal{B}_1} \mathcal{B} \cong \mathcal{E}$. Conversely, a Fell bundle correspondence with $\mathcal{E} = \mathcal{B}$ as right Hilbert $\mathcal{B}$-module has $\mathcal{E}_1 = \mathcal{B}_1$ as a right Hilbert $\mathcal{B}_1$-module.

A Fell bundle correspondence with $\mathcal{E} = \mathcal{B}$ as a right Hilbert $\mathcal{B}$-module is the same as a continuous multiplication $\mathcal{A} \times \mathcal{B} \to \mathcal{B}$ with the properties in Definition 2.13. These properties are equivalent to f being a morphism $\mathcal{A} \to \mathcal{M}(\mathcal{B})$ of Fell bundles over $\mathcal{C}$ as defined in the statement of the proposition. The equivalence of joint and separate continuity follows because $\|f(a)\| \leq \|a\|$ for all $a \in \mathcal{A}$. □

Finally, we specialise to representations of $\mathcal{C}$-actions. These are, by definition, transformations to a C^*-algebra B equipped with the trivial $\mathcal{C}$-action. The trivial $\mathcal{C}$-action on B corresponds to the constant Fell bundle with fibre B and $\upsilon_h^B = 1_B$ for all $h \in H$. A morphism of Fell bundles from $(\mathcal{A}, \upsilon)$ to such a constant Fell bundle is a map $\rho\colon \mathcal{A} \to \mathcal{M}(B)$ such that

(1) ρ restricts to linear maps $\mathcal{A}_g \to \mathcal{M}(B)$ for all $g \in G$;
(2) $\rho(a^*) = \rho(a)^*$ and $\rho(a_1 \cdot a_2) = \rho(a_1) \cdot \rho(a_2)$ for all $a, a_1, a_2 \in \mathcal{A}$;
(3) for each $b \in B$, the map $a \mapsto \rho(a)b$ is continuous from $\mathcal{A}$ to B (continuity);
(4) $\rho(\mathcal{A}_1) \cdot \mathcal{B}_1 = \mathcal{B}_1$ or equivalently $\rho(\mathcal{A}_1) \cdot \mathcal{B} = \mathcal{B}$ (nondegeneracy);
(5) $\rho(\upsilon_h^{\mathcal{A}}) = 1$ for all $h \in H$;

the last condition involves the canonical extension of a representation of a Fell bundle to multipliers. Except for the last condition, this is the standard definition of a representation of a Fell bundle (see [3]).

We are going to describe the crossed product of a $\mathcal{C}$-action by correspondences as a quotient of the cross-sectional C^*-algebra of the associated Fell bundle, generalising the description in [1] for strict $\mathcal{C}$-actions. The cross-sectional C^*-algebra $\mathrm{C}^*(\mathcal{A})$ of a Fell bundle $\mathcal{A}$ over the locally compact group G is constructed in [3, Chapter VIII] as the C^*-completion of the space $\mathrm{C_c}(\mathcal{A})$ of compactly supported sections of $\mathcal{A}$ with a suitable convolution and involution. It comes with a canonical map $\rho^{\mathrm{u}}\colon \mathcal{A} \to \mathcal{M}(\mathrm{C}^*(\mathcal{A}))$, which is a representation of $\mathcal{A}$ viewed as a G-C^*-algebra (via Theorem 2.11).

Definition 2.16. For a Fell bundle $(\mathcal{A}, \upsilon)$ over $\mathcal{C}$, let $\mathcal{I}_\upsilon$ be the closed, two-sided ideal of $\mathrm{C}^*(\mathcal{A})$ generated by the multipliers $\{\rho^{\mathrm{u}}(\upsilon_h) - 1 : h \in H\}$, that is,

$$\mathcal{I}_\upsilon := \overline{\operatorname{span}}\{x(\rho^{\mathrm{u}}(\upsilon_h) - 1)y : x, y \in \mathrm{C}^*(\mathcal{A}), h \in H\}.$$

We call $\mathrm{C}^*(\mathcal{A}, \upsilon) := \mathrm{C}^*(\mathcal{A})/\mathcal{I}_\upsilon$ the *cross-sectional C^*-algebra* of $(\mathcal{A}, \upsilon)$.

Since $\rho(\upsilon_h) = 1$ is the only extra condition needed for a representation of $\mathcal{A}$ to be a representation of $(\mathcal{A}, \upsilon)$, $\mathrm{C}^*(\mathcal{A}, \upsilon)$ is the largest quotient of $\mathrm{C}^*(\mathcal{A})$ on which ρ^{u} gives a representation of $(\mathcal{A}, \upsilon)$ as a $\mathcal{C}$-C^*-algebra.

Proposition 2.17. *The C^*-algebra $\mathrm{C}^*(\mathcal{A}, \upsilon) := \mathrm{C}^*(\mathcal{A})/\mathcal{I}_\upsilon$ with the canonical representation of $\mathcal{A}$ is a crossed product $(\mathcal{A}, \upsilon) \rtimes \mathcal{C}$. That is, representations of $\mathrm{C}^*(\mathcal{A}, \upsilon)$ correspond bijectively to representations of $(\mathcal{A}, \upsilon)$.*

Proof. Morphisms $\mathrm{C}^*(\mathcal{A}) \to B$ are in bijection with representations of the Fell bundle $\mathcal{A}$ on B, mapping a representation ρ of $\mathcal{A}$ to its integrated form $\textstyle\int\rho$ (see [3, VIII.11]). Representations of $\mathrm{C}^*(\mathcal{A}, \upsilon) = \mathrm{C}^*(\mathcal{A})/\mathcal{I}_\upsilon$ correspond bijectively to representations of $\mathrm{C}^*(\mathcal{A})$ that vanish on $\mathcal{I}_\upsilon$. Moreover, for a representation π of $\mathcal{A}$,

$$\begin{aligned} \textstyle\int\rho(\mathcal{I}_\upsilon) = 0 &\iff \textstyle\int\rho(\rho^{\mathrm{u}}(\upsilon_h)\xi - \xi) = 0 \text{ for all } h \in H,\ \xi \in \mathrm{C_c}(\mathcal{A}) \\ &\iff \rho(\upsilon_h \cdot a) = \rho(a) \text{ for all } a \in \mathcal{A}. \end{aligned}$$

Thus the bijection from representations of $\mathcal{A}$ on B to morphisms $\mathrm{C}^*(\mathcal{A}) \to B$ restricts to a bijection from representations of $(\mathcal{A}, \upsilon)$ to morphisms $\mathrm{C}^*(\mathcal{A}, \upsilon) \to B$. □

3. Equivalence of crossed modules

One should expect equivalent crossed modules to have equivalent 2-categories of actions on C^*-algebras by correspondences. What does this mean? A functor $\mathfrak{Corr}(\mathcal{C}) \to \mathfrak{Corr}(\mathcal{C}')$ for two crossed modules $\mathcal{C}$ and $\mathcal{C}'$ consists of

- a map F on objects, which maps each $\mathcal{C}$-action x to a $\mathcal{C}'$-action $F(x)$;
- for any two $\mathcal{C}$-actions x_1 and x_2, a functor

$$F\colon \mathfrak{Corr}_{\mathcal{C}}(x_1, x_2) \to \mathfrak{Corr}_{\mathcal{C}'}(F(x_1), F(x_2));$$

- for any three $\mathcal{C}$-actions x_1, x_2, x_3, natural transformations

$$\mathfrak{Corr}_{\mathcal{C}}(x_2, x_3) \times \mathfrak{Corr}_{\mathcal{C}}(x_1, x_2) \to \mathfrak{Corr}_{\mathcal{C}'}(F(x_1), F(x_3)), \qquad F(f) \circ F(g) \cong F(f \circ g).$$

- natural isomorphisms $F(1_x) \cong 1_{F(x)}$;

the natural transformations in the last two conditions must satisfy suitable coherence axioms (see [9]). Such a functor is an equivalence if the map F on objects is essentially surjective and the functors on the arrow groupoids are equivalences.

An equivalence of 2-categories has a quasi-inverse that is again a functor, such that the compositions in either order are equivalent to the identity functor in a suitable sense.

We will only consider functors constructed from homomorphisms of crossed modules. Let $\mathcal{C}_i = (G_i, H_i, \partial_i, c_i)$ for $i = 1, 2$ be crossed modules. A *homomorphism of crossed modules* $\mathcal{C}_1 \to \mathcal{C}_2$ is a pair of continuous group homomorphisms $\varphi\colon G_1 \to G_2$, $\psi\colon H_1 \to H_2$ such that $\partial_2 \circ \psi = \varphi \circ \partial_1$ and $c_{2,\varphi(g)}(\psi(h)) = \psi(c_{1,g}(h))$. Such a homomorphism induces a functor

$$(\varphi, \psi)^*\colon \mathfrak{Corr}(\mathcal{C}_2) \to \mathfrak{Corr}(\mathcal{C}_1), \tag{3.1}$$

by sending a $\mathcal{C}_2$-action $(A, \alpha, \omega, \eta)$ to $(A, \varphi^*\alpha, (\varphi \times \varphi)^*\omega, \psi^*\eta)$, a transformation $(\mathcal{E}, \chi)$ to $(\mathcal{E}, \varphi^*\chi)$, and a modification W again to W; the natural transformations in the definition of a functor above are trivial and therefore coherent.

When we translate to Fell bundles using Theorem 2.11, the functor (3.1) sends a Fell bundle $(\mathcal{A}, \upsilon)$ over $\mathcal{C}_1$ to the Fell bundle $(\varphi^*\mathcal{A}, \psi^*\upsilon)$ over $\mathcal{C}_2$, where $\varphi^*\mathcal{A}$ denotes the pull-back of $\mathcal{A}$ along φ and $\psi^*\upsilon = \upsilon \circ \psi$.

Definition 3.2. The *arrow groupoid* of $\mathcal{C} = (G, H, \partial, c)$ is the transformation groupoid $H \ltimes G$ for the right action of the topological group H on the topological space G defined by $g \cdot h := g\partial(h)$.

A homomorphism (φ, ψ) of crossed modules induces a functor between the arrow groupoids. We call (φ, ψ) an *equivalence* if the induced functor on arrow groupoids gives an equivalence of topological groupoids.

To make this precise, we turn a functor between the arrow groupoids into a Hilsum–Skandalis morphism, that is, a topological space endowed with commuting actions of $H_1 \ltimes G_1$ and $H_2 \ltimes G_2$ (see [8]). Let $X := G_1 \times H_2$ and define anchor maps $\pi_i\colon X \to G_i$ for $i = 1, 2$ by $\pi_1(g_1, h_2) := g_1$ and $\pi_2(g_1, h_2) := \varphi(g_1)\partial_2(h_2)$. Define a left H_1-action and a right H_2-action on X by

$$\begin{aligned} h \cdot (g_1, h_2) &:= (g_1\partial_1(h)^{-1}, \psi(h)h_2) &&\text{for } h \in H_1,\\ (g_1, h_2) \cdot h &:= (g_1, h_2 h) &&\text{for } h \in H_2. \end{aligned}$$

These definitions turn X into a $H_1 \ltimes G_1$-$H_2 \ltimes G_2$-bispace. This bispace is always a Hilsum–Skandalis morphism, that is, the action of $H_2 \ltimes G_2$ is free and proper and π_1 induces a homeomorphism from the $H_2 \ltimes G_2$-orbit space to G_1. A functor is an equivalence if the associated bispace is a Morita equivalence as in [13].

In our case, the homomorphism (φ, ψ) is an equivalence if and only if the action of $H_1 \ltimes G_1$ on X is free and proper with orbit space projection π_2. Equivalently, X is a Morita equivalence between $H_1 \ltimes G_1$ and $H_2 \ltimes G_2$ as in [13]. It is enough to check that H_1 acts freely and properly on X with $X/H_1 \cong G_2$ via π_2.

Lemma 3.3. *The homomorphism (φ, ψ) is an equivalence if and only if*

(1) *the map*
$$(\partial_1, \psi)\colon H_1 \to \{(g_1, h_2) \in G_1 \times H_2 : \varphi(g_1) = \partial_2(h_2)\} = G_1 \times_{G_2} H_2$$
is a homeomorphism, where the codomain carries the subspace topology;

(2) *the map $\pi_2\colon G_1 \times H_2 \to G_2$, $(g_1, h_2) \mapsto \varphi(g_1) \cdot \partial_2(h_2)$, is an open surjection.*

The first condition says that H_1 is the pull-back of H_2 along φ, the second condition is a transversality condition for this pull-back.

Proof. The freeness of the H_1-action on X means that no $h \neq 1$ in H_1 has $\psi(h) = 1$ and $\partial_1(h) = 1$, that is, the map in (1) is injective.

Let $(g_1, h_2) \in G_1 \times H_2$ and $(g_1', h_2') \in G_1 \times H_2$. They have the same π_2-image if and only if $\varphi(g_1)\partial_2(h_2) = \varphi(g_1')\partial_2(h_2')$, if and only if $((g_1')^{-1}g_1, h_2'h_2^{-1}) \in G_1 \times H_2$ satisfies $\varphi((g_1')^{-1}g_1) = \partial_2(h_2'h_2^{-1})$; hence the surjectivity of the map in (1) means that the map $\pi_2'\colon X/H_1 \to G_2$ induced by π_2 is injective. The map π_2' is automatically continuous, and it is open if and only if π_2 is because the projection $X \to X/H_1$ is open and continuous. Hence condition (2) means that π_2' is surjective and open. Being injective and continuous as well, it is a homeomorphism.

Finally, the properness of the H_1-action is equivalent to the following: for all compact subsets $K \subseteq G_1$, $L \subseteq H_2$, the set of $h \in H_1$ with $\partial_1(h) \in K^{-1} \cdot K$ and $\psi(h) \in L \cdot L^{-1}$ is compact. This is equivalent to the properness of the map in (1). We already know that this map is a continuous bijection. Such a map is proper if and only if it is a homeomorphism. $\square$

For discrete crossed modules, [16, Proposition 6.3] says that the equivalences in the above sense are the acyclic cofibrations in a suitable model structure.

The arrow groupoid does not yet encode the multiplication in G and the conjugation action c. These are encoded in a continuous functor

$$
\begin{aligned}
(3.4)\qquad M\colon (H\ltimes G)\times(H\ltimes G)\to(H\ltimes G),\qquad &(g_1,g_2)\mapsto g_1\cdot g_2,\\
&(g_1,h_1),(g_2,h_2)\mapsto (g_1\cdot g_2, c_{g_2}^{-1}(h_1)\cdot h_2);
\end{aligned}
$$

here (g,h) denotes the 2-arrow $h\colon g\Rightarrow g\partial(h)$. The existence and associativity of this functor is equivalent to the axioms of a crossed module. The orbit space of the arrow groupoid is $\pi_1(\mathcal{C}) := \operatorname{coker}\partial$; the group structure on the orbit space is induced by M; the isotropy group of any $g\in G$ is isomorphic to $\pi_2(\mathcal{C}) := \ker\partial$. This is an Abelian group, and the action c induces a $\pi_1(\mathcal{C})$-module structure on it; this module structure may also be expressed through M. Since we may express them through canonical extra structure on the arrow groupoid, the group $\pi_1(\mathcal{C})$ and the $\pi_1(\mathcal{C})$-module $\pi_2(\mathcal{C})$ are invariant under equivalences; for crossed modules of locally compact groups, this invariance includes the induced topologies on them.

Discrete crossed modules are classified up to equivalence by the group $\pi_1(\mathcal{C})$, the $\pi_1(\mathcal{C})$-module $\pi_2(\mathcal{C})$, and a cohomology class in $\mathrm{H}^3(\pi_1(\mathcal{C}),\pi_2(\mathcal{C}))$ (see [10]). We shall not attempt such a complete classification of crossed modules of locally compact groups here. It is useful, however, to know that π_1 and π_2 are invariant under equivalence.

Theorem 3.5. *The functor $\mathfrak{Corr}(\mathcal{C}_2)\to\mathfrak{Corr}(\mathcal{C})$ induced by an equivalence of crossed modules $(\varphi,\psi)\colon\mathcal{C}\to\mathcal{C}_2$ is an equivalence of 2-categories.*

This equivalence intertwines the crossed product functors on both categories.

Proof. We must show that any $\mathcal{C}$-action (A,α,ω,η) is isomorphic to

$$(\varphi,\psi)^*(A_2,\alpha_2,\omega_2,\eta_2)$$

for an action of $\mathcal{C}_2$ and that any transformation $(\mathcal{E},\chi)$ between $\mathcal{C}$-actions is isomorphic to $(\varphi,\psi)^*(\mathcal{E}_2,\chi_2)$ for a transformation of $\mathcal{C}_2$-actions $(\mathcal{E}_2,\chi_2)$ that is unique up to isomorphism. Since $(\varphi,\psi)^*$ does not change the underlying C^*-algebras, we may put $A_2 := A$.

Define isomorphisms of correspondences $\eta_{g,h}\colon\alpha_g\to\alpha_{g\partial(h)}$ for $g\in G$, $h\in H$ by

$$\eta_{g,h}\colon\alpha_g\cong A\otimes_A\alpha_g\xrightarrow{\eta_h\otimes_A\mathrm{Id}_{\alpha_g}}\alpha_{\partial(h)}\otimes_A\alpha_g\xrightarrow{\omega(g,\partial(h))}\alpha_{g\partial(h)}.$$

The space $G\times H$ is the space of 2-arrows in $\mathcal{C}$, and an action in the sense of [2, Definition 4.1] provides $\eta_{g,h}$ as above. We removed redundancy in Definition 2.1 and kept only $\eta_{1,h}=\eta_h$. Now we need the whole family $\eta_{g,h}$. It depends continuously on $(g,h)\in G\times H$. The naturality of the isomorphisms ω_{g_1,g_2} with respect to 2-arrows says that the diagrams

$$
(3.6)\qquad
\begin{tikzcd}
\alpha_{g_2}\otimes_A\alpha_{g_1} \arrow[r, "\omega_{g_1,g_2}"] \arrow[d, "\eta_{g_2,h_2}\otimes_A\eta_{g_1,h_1}"'] & \alpha_{g_1g_2} \arrow[d, "\eta_{g_1g_2,c_{g_2}^{-1}(h_1)h_2}"] \\
\alpha_{g_2\partial(h_2)}\otimes_A\alpha_{g_1\partial(h_1)} \arrow[r, "\omega_{g_1\partial(h_1),g_2\partial(h_2)}"'] & \alpha_{g_1\partial(h_1)\cdot g_2\partial(h_2)}
\end{tikzcd}
$$

commute for all $g_1,g_2\in G$, $h_1,h_2\in H$.

The isomorphisms $\eta_{g,h}\colon\alpha_g\to\alpha_{g\partial(h)}$ for $g\in G$, $h\in H$ turn α into a $H\ltimes G$-equivariant Hilbert bimodule over $\mathrm{C}_0(G,A)$. Since $H\ltimes G$ is Morita equivalent to $G_2\rtimes H_2$ via the bispace X, we may transport this to a $G_2\rtimes H_2$-equivariant Hilbert bimodule α_2 over $\mathrm{C}_0(G_2,A)$. We describe α_2 more explicitly.

The pull-back $\pi_1^*\alpha\cong\mathrm{C}_0(H_2)\otimes\alpha$ along $\pi_1\colon X\to G$ is an $H\times H_2$-equivariant Hilbert module over $\mathrm{C}_0(X,A)$. Let $\alpha_2\subseteq\mathcal{M}(\pi_1^*\alpha)$ be the space of all bounded continuous sections of $\pi_1^*\alpha$ that are H-invariant and whose norm function is in

$\mathrm{C}_0(X/H)$. Since $X/H \cong G_2$ via π_2 and the actions of H_2 and H commute, we may view this as an H_2-equivariant Hilbert $\mathrm{C}_0(G_2, A)$-module.

The left A-action survives these constructions because the action of H commutes with it. The fibre of α_2 at $g_2 \in G_2$ is isomorphic to α_g for any $g \in G$ for which there is $h_2 \in H_2$ with $\pi_2(g, h_2) = g_2$, that is, $g_2 = \varphi(g)\partial_2(h_2)$. These isomorphisms on the fibres are continuous and give a canonical H-equivariant isomorphism $\varphi^*\alpha_2 \cong \alpha$.

The pull-backs $\pi_1^*\alpha$, $\pi_2^*\alpha$ and $\mu^*\alpha$ in the definition of ω are $H \times H$-equivariant Hilbert modules over $\mathrm{C}_0(G \times G, A)$. That is, they are representations of the groupoid $(H \ltimes G)^2 = (H \times H) \ltimes (G \times G)$. This groupoid is equivalent to $(H_2 \ltimes G_2)^2$ via the bispace X^2. Now pull back $\pi_1^*\alpha$, $\pi_2^*\alpha$, $\mu^*\alpha$ and the isomorphism $\omega\colon \pi_2^*\alpha \otimes_{\mathrm{C}_0(G\times G,A)} \pi_1^*\alpha \to \mu^*\alpha$ to X^2 and push all this down to the category of H_2^2-equivariant Hilbert $\mathrm{C}_0(G_2^2, A)$-modules by taking H^2-invariants. The resulting Hilbert $\mathrm{C}_0(G_2^2, A)$-modules are canonically isomorphic to $\pi_1^*\alpha_2$, $\pi_2^*\alpha_2$, and $\mu^*\alpha_2$, respectively. Hence ω induces a $\mathrm{C}_0(G_2 \times G_2)$-linear $H_2 \times H_2$-equivariant unitary operator

$$\omega_2\colon \pi_2^*\alpha_2 \otimes_{\mathrm{C}_0(G_2\times G_2,A)} \pi_1^*\alpha_2 \to \mu^*\alpha_2.$$

The $H_2 \times H_2$-equivariance shows that ω_2 satisfies the analogue of (3.6), which gives conditions (2) and (3) in Definition 2.1. To prove that ω_2 inherits condition (1) in Definition 2.1, we use that for each $g_1, g_2, g_3 \in G_2$ there are $g_1', g_2', g_3' \in G$ such that $\omega_{2,g_1,g_2} = \omega_{g_1',g_2'}$, $\omega_{2,g_1g_2,g_3} = \omega_{g_1'g_2',g_3'}$, $\omega_{2,g_2,g_3} = \omega_{g_2',g_3'}$, $\omega_{2,g_1,g_2g_3} = \omega_{g_1',g_2'g_3'}$ with suitable natural identifications of the fibres $\alpha_{2,g_i} \cong \alpha_{g_i'}$ and $\alpha_{2,g_ig_j} \cong \alpha_{g_i'g_j'}$. We also find a canonical isomorphism $(\varphi \times \varphi)^*\omega_2 \cong \omega$. Hence $(\varphi, \psi)^*(A, \alpha_2, \omega_2, \eta_2) = (A, \alpha, \omega, \eta)$.

In the same way, we may also lift a transformation $(\mathcal{E}, \chi)$ of $\mathcal{C}$-actions to one between the corresponding $\mathcal{C}_2$-actions. We may put $\mathcal{E}_2 = \mathcal{E}$ because our lifting does not change the underlying C^*-algebras. The assumptions in Definition 2.3 imply that the isomorphism χ is H-equivariant. Hence we may pull it back to an $H \times H_2$-equivariant isomorphism over X and then push down to G_2 to get an H_2-equivariant isomorphism $\chi_2\colon \mathcal{E} \otimes_B \beta_2 \to \alpha_2 \otimes_A \mathcal{E}$ over G_2. The same arguments as above show that $(\mathcal{E}, \chi_2)$ is a transformation that is a $(\varphi, \psi)^*$-preimage of the transformation $(\mathcal{E}, \chi)$, and the only one with this property up to isomorphism of transformations.

It is clear that $(\varphi, \psi)^*$ maps a trivial action of $\mathcal{C}_2$ to a trivial action of $\mathcal{C}$. Furthermore, on the level of transformations, $\mathcal{E}$ is a morphism (its underlying Hilbert module is B itself) if and only if $(\varphi, \psi)^*\mathcal{E}$ is a morphism. Therefore, $(\varphi, \psi)^*$ gives a bijection between $\mathcal{C}_2$-representations of $(A, \alpha_2, \omega_2, \eta_2)$ on B and $\mathcal{C}$-representations of $(A, \alpha, \omega, \eta)$ on B. By the universal property, there is a unique isomorphism $A \rtimes_{\alpha,\omega,\eta} \mathcal{C} \cong A \rtimes_{\alpha_2,\omega_2,\eta_2} \mathcal{C}_2$ that maps the universal $\mathcal{C}$-representation to the universal $\mathcal{C}_2$-representation. □

Example 3.7. Let $\mathcal{C} = (G, H, \partial, c)$ be a crossed module and let $G_1 \subseteq G$ be a closed subgroup such that the map $G_1 \times H \to G$, $(g, h) \mapsto g \cdot \partial(h)$, is open and surjective. Let $H_1 := \partial^{-1}(G_1)$ and let $\partial_1\colon H_1 \to G_1$ and $c_1\colon G_1 \to \mathrm{Aut}(H_1)$ be the restrictions of ∂ and c. Then the embedding of $\mathcal{C}_1 = (G_1, H_1, \partial_1, c_1)$ into $\mathcal{C}$ is an equivalence by Lemma 3.3. The second condition in Lemma 3.3 is our assumption. The first condition is that the group homomorphism

$$H_1 \to \{(g_1, h) \in G_1 \times H : \varphi(g_1) = \partial(h)\}, \qquad h_1 \mapsto (\psi(h_1), \partial_1(h_1)),$$

is a homeomorphism. Indeed, the map has the restriction of the first coordinate projection as a continuous inverse.

Theorem 3.5 says that a Fell bundle over $\mathcal{C}_1$ extends to one over $\mathcal{C}$ in a natural and essentially unique way.

Example 3.8. Let $N \subseteq H$ be a closed $c(G)$-invariant subgroup such that ∂ restricts to a homeomorphism from N onto a closed subgroup of G; then $\partial(N)$ is normal in G. Let $G_2 := G/\partial(N)$, $H_2 := H/N$, and let $\partial_2 \colon H_2 \to G_2$ and $c_2 \colon G_2 \to \mathrm{Aut}(H_2)$ be the induced maps. Then the projection map from $\mathcal{C}$ to $\mathcal{C}_2 = (G_2, H_2, \partial_2, c_2)$ is an equivalence by Lemma 3.3. The second condition in Lemma 3.3 follows because already the projection $G \to G_2$ is open and surjective. The first condition requires the map

$$H \to \{(g, h_2) \in G \times H_2 : \varphi(g) = \partial_2(h_2)\}, \qquad h \mapsto (\psi(h), \partial(h)),$$

to be a homeomorphism. Injectivity and surjectivity of this map are clear, and its properness is not hard to check. This implies that the map is a homeomorphism.

Theorem 3.5 says that any Fell bundle over $\mathcal{C}$ is the pull-back of a Fell bundle over $\mathcal{C}_2$, which is unique up to isomorphism and depends naturally on the original Fell bundle over $\mathcal{C}$. The reason to expect this is that the unitary multipliers υ_h for $h \in N$ trivialise our Fell bundle over N-cosets.

Example 3.9. Even more specially, assume that $\mathcal{C}$ is group-like, that is, ∂ is a homeomorphism onto a closed (normal) subgroup of G. Theorem 3.5 says that $\mathcal{C}$-actions are equivalent to actions of the locally compact group $G/\partial(H)$ viewed as a crossed module. Actions of the latter are equivalent to Fell bundles over the locally compact group $G/\partial(H)$ in the usual sense. Actions of $\mathcal{C}$ are a Fell bundle analogue of Green twisted actions. In particular, any Green twisted action of $(G, \partial(H))$ gives rise to a Fell bundle over $G/\partial(H)$.

For discrete groups, this equivalence is already contained in [5].

We have defined when a crossed module homomorphism is an equivalence. Since its inverse is usually not described by a crossed module homomorphism, we are led to the following equivalence relation for crossed modules:

Definition 3.10. Two crossed modules of locally compact groups $\mathcal{C}$ and $\mathcal{C}'$ are *equivalent* if they are connected by a chain of crossed module homomorphisms

$$\mathcal{C} = \mathcal{C}_0 \leftarrow \mathcal{C}_1 \to \mathcal{C}_2 \leftarrow \mathcal{C}_3 \to \dots \leftarrow \mathcal{C}'$$

where each arrow is a crossed module homomorphism that is an equivalence.

Since equivalence of 2-categories formulated in terms of functors is a symmetric relation (quasi-inverses exist and are again functors), equivalent crossed modules have equivalent action 2-categories $\mathfrak{Corr}(\mathcal{C})$ by Theorem 3.5. The topological group $\pi_1(\mathcal{C})$ and the topological $\pi_1(\mathcal{C})$-module $\pi_2(\mathcal{C})$ are invariant under equivalence because, as already mentioned, equivalences implemented by homomorphisms preserve π_1 and π_2.

3.1. **Simplification of thin crossed modules.** We call a crossed module $\mathcal{C} = (G, H, \partial, c)$ *thin* if ∂ is injective and has dense range. In the discrete case, this implies that $\mathcal{C}$ is equivalent to the trivial crossed module. An interesting example is the crossed module associated to a dense embedding $\mathbb{Z} \to \mathbb{T}$, which acts on the corresponding noncommutative torus (see [1]). Being thin is an invariant of equivalence of crossed modules. In a thin crossed module, the action c is dictated by $\partial(c_g(h)) = g\partial(h)g^{-1}$ because ∂ is injective.

We want to simplify thin crossed modules up to equivalence. The best result is available if both G and H are Lie groups (we allow an arbitrary number of connected components). In that case, the following theorem gives a complete classification up to equivalence.

Theorem 3.11. *Any thin crossed module of Lie groups is equivalent to one where $G = \mathbb{R}^n$ for some $n \in \mathbb{N}$ and H is a dense subgroup of G with the discrete topology.*

And it is equivalent to one where $G = \mathbb{T}^n$ for some $n \in \mathbb{N}$ and H is a dense subgroup of G with the discrete topology.

Two thin crossed modules $\mathcal{C}_i = (G_i, H_i, \partial_i, c_i)$ with $G_i = \mathbb{R}^{n_i}$ for $i = 1, 2$ and discrete H_i are equivalent if and only if there is an invertible linear map $\mathbb{R}^{n_1} \to \mathbb{R}^{n_2}$ mapping $\partial_1(H_1)$ onto $\partial_2(H_2)$.

Proof. Let $\mathcal{C} = (G, H, \partial, c)$ be a thin crossed module of Lie groups. Let $G_1 \subseteq G$ be the connected component of the identity. Let $H_1 = \partial^{-1}(G_1) \subseteq H$. Since $\partial(H)$ is dense in G, it meets every connected component. Hence we are in the situation of Example 3.7, and $\mathcal{C}$ is equivalent to $\mathcal{C}_1 = (G_1, H_1, \partial_1, c_1)$, where ∂_1 and c_1 are the restrictions of ∂ and c. The Lie group G_1 is connected.

Let $\varphi\colon G_2 \to G_1$ be the universal covering of G_1 and let

$$H_2 := \{(h_1, g_2) \in H_1 \times G_2 : \partial_1(h_1) = \varphi(g_2)\}$$

be its pull-back to a covering of H_1. Let $\psi\colon H_2 \to H_1$ and $\partial_2\colon H_2 \to G_2$ be the coordinate projections. Then ψ is a covering map; its kernel is a discrete central subgroup $N \subseteq H_2$. The homomorphism ∂_2 maps N homeomorphically onto the kernel of the covering map φ. Hence we are in the situation of Example 3.8, and $\mathcal{C}_1$ is equivalent to $\mathcal{C}_2 = (G_2, H_2, \partial_2, c_2)$, where c_2 lifts c_1. (The composition $\mathcal{C}_2 \to \mathcal{C}_1 \to \mathcal{C}$ is also an equivalence, so we could have gone to $\mathcal{C}_2$ in only one step.)

The Lie group G_2 is simply connected. Let N_2 be the connected component of the identity in H_2. We claim that ∂_2 maps N_2 homeomorphically onto a closed subgroup of G_2. We are going to prove this later, so let us assume this for a moment. Then we are again in the situation of Example 3.8. Letting $G_3 := G_2/\partial_2(N_2)$, $H_3 := H_2/N_2$, and taking the induced maps ∂_3 and c_3, we get a crossed module equivalence $\mathcal{C}_2 \to \mathcal{C}_3 = (G_3, H_3, \partial_3, c_3)$.

The Lie group H_3 is discrete, and G_3 is still a simply connected Lie group because it is a quotient of a simply connected group by a connected normal subgroup. The conjugation action $c_3\colon G_3 \to \operatorname{Aut}(H_3)$ is a continuous action of a connected group on a discrete space. It must be trivial. Hence $g\partial_3(h)g^{-1} = \partial_3(h)$ for all $h \in H_3$. Since we started with a thin crossed module, $\partial_3(H_3)$ is dense in G_3, so the above equality extends to all of G_3, proving that G_3 is Abelian. Any Abelian simply connected Lie group is of the form $\mathbb{R}^n$ for some $n \in \mathbb{N}$. Thus $\mathcal{C}_3$ is a crossed module equivalent to $\mathcal{C}$ with $G_3 = \mathbb{R}^n$ and discrete H_3.

Now we show that ∂_2 is a homeomorphism from N_2 onto a closed subgroup of G_2 if G_2 is simply connected. Let $\mathfrak{h}$ and $\mathfrak{g}$ be the Lie algebras of H_2 and G_2, respectively. The dense embedding ∂_2 induces an injective map $\mathfrak{h} \to \mathfrak{g}$, whose image is a Lie ideal. Hence there is a Lie algebra $\mathfrak{g}/\mathfrak{h}$ and a Lie algebra homomorphism $\mathfrak{g} \to \mathfrak{g}/\mathfrak{h}$. By Lie's theorems, there is a simply connected Lie group K with Lie algebra $\mathfrak{g}/\mathfrak{h}$ and a Lie group homomorphism $G_2 \to K$ that induces $\mathfrak{g} \to \mathfrak{g}/\mathfrak{h}$ on the Lie algebras. The kernel of this homomorphism is a connected, closed, normal subgroup, and its Lie algebra is $\mathfrak{h}$. This closed normal subgroup is $\partial_2(N_2)$ because the exponential map is a local homeomorphism from the Lie algebra to the Lie group near the identity element. Moreover, the map ∂_2 is indeed a homeomorphism from N_2 onto this closed normal subgroup.

To get another equivalent crossed module $\mathcal{C}_4$ with $G_4 \cong \mathbb{T}^n$, we choose elements $h_1, \dots, h_n \in H_3$ whose images form a basis in $\mathbb{R}^n$, using the density of $\partial(H_3)$ in $G_3 = \mathbb{R}^n$. These elements generate a subgroup N_3 isomorphic to $\mathbb{Z}^n$ in H_3, which is mapped by ∂_3 onto a discrete subgroup in G_3. Using Example 3.8 once again, we find that $\mathcal{C}_3$ is equivalent to $\mathcal{C}_4$ with $G_4 = G_3/N_3 \cong \mathbb{T}^n$, $H_4 = H_3/N_3$ and ∂_4 and c_4 induced by ∂_3 and c_3. The quotient H_4 is still discrete, and mapped by ∂_4 onto a dense subgroup in the torus G_4. Thus $\mathcal{C}_4$ has the desired form.

Finally, we show that crossed modules with $G_i \cong \mathbb{R}^{n_i}$ and discrete H_i are only equivalent when they are isomorphic through some invertible linear map $G_1 \to G_2$ that maps $\partial(H_1)$ onto $\partial(H_2)$.

We observe first that the quotient Lie algebra $\mathfrak{g}/\mathfrak{h}$ for a crossed module $\mathcal{C}$ is invariant under equivalence. Any homomorphism of crossed modules $\mathcal{C}_1 \to \mathcal{C}_2$ induces a pair of Lie algebra homomorphisms $\mathfrak{h}_1 \to \mathfrak{h}_2$ and $\mathfrak{g}_1 \to \mathfrak{g}_2$ that intertwine the differentials of ∂_1 and ∂_2 and hence induce a homomorphism $\mathfrak{g}_1/\mathfrak{h}_1 \to \mathfrak{g}_2/\mathfrak{h}_2$. It is not hard to see that this map is invertible if the homomorphism is an equivalence. Roughly speaking, $\mathfrak{g}/\mathfrak{h}$ is the tangent space at the unit element in $\pi_1(\mathcal{C})$, and it is invariant because $\pi_1(\mathcal{C})$ as a topological group is invariant.

If $X \in \mathfrak{g}/\mathfrak{h}$ and one representative $\hat{X} \in \mathfrak{g}$ has the property that $\exp(\hat{X}) \in \partial(H)$, then this holds for all representatives. Let us denote this subset of $\mathfrak{g}/\mathfrak{h}$ by $T(\mathcal{C})$. A crossed module homomorphism $\mathcal{C}_1 \to \mathcal{C}_2$ maps $T(\mathcal{C}_1) \to T(\mathcal{C}_2)$, and if it is an equivalence, then it maps $T(\mathcal{C}_1)$ onto $T(\mathcal{C}_2)$ because it induces a bijection between $G_i/\partial_i(H_i)$. Hence the pair consisting of the Lie algebra $\mathfrak{g}/\mathfrak{h}$ and the subset $T(\mathcal{C})$ is invariant under equivalence of crossed modules (in the sense that equivalent crossed modules have isomorphic invariants).

If a crossed module has discrete H and $G = \mathbb{R}^n$, then we identify $\mathfrak{g}/\mathfrak{h} = \mathbb{R}^n$ in the obvious way and find that the subset of X with $\exp(\hat{X}) \in \partial(H)$ is precisely $\partial(H)$. This determines our crossed module because ∂ is injective, c is trivial, and H is discrete. Hence crossed modules of this form are only equivalent when they are isomorphic. □

Given a thin crossed module $\mathcal{C} = (G, H, \partial, c)$, the proof of Theorem 3.11 shows that $\mathfrak{g}/\mathfrak{h}$ is the Abelian Lie algebra $\mathbb{R}^n$ for some $n \in \mathbb{N}$, that $T(\mathcal{C}) \subseteq \mathfrak{g}/\mathfrak{h}$ is a dense subgroup, and that $\mathcal{C}$ is equivalent to the thin Abelian crossed module $\mathcal{C}'$ with $G' = \mathfrak{g}/\mathfrak{h}$, $H' = T(\mathcal{C})$ with the discrete topology, ∂' the inclusion map, and trivial c'. Hence we get an explicit Abelian replacement for $\mathcal{C}$ and do not have to follow the steps in the above proof to construct it.

Example 3.12. Consider the thin crossed module from a dense embedding $\mathbb{R} \subset \mathbb{T}^2$ given by a line of irrational slope. This is not yet in standard form because $\mathbb{R}$ is not discrete. Of course, the image of the connected component of the identity is not closed in this example. But when we pass to the universal covering $\mathbb{R}^2$ of $\mathbb{T}^2$, then we get the equivalent crossed module $\mathbb{R} + \mathbb{Z}^2 \subset \mathbb{R}^2$. Now the image of the connected component is a line of irrational slope, which is closed in $\mathbb{R}^2$, and we may divide it out to get an equivalent crossed module $\mathbb{Z}^2 \to \mathbb{R}$. This is now in standard form, and indeed of the expected form $G' = \mathfrak{g}/\mathfrak{h}$, $H' = T(\mathcal{C})$.

Example 3.13. Let $G = \mathbb{T}$ and let $H = G$ with the discrete topology. This is an example of a crossed module which is already in the second standard form of Theorem 3.11. Theorem 3.11 works also if our Lie groups have uncountably many components.

The standard form with $G = \mathbb{T}^n$ in Theorem 3.11 is sometimes less useful because it is less unique, but it has the advantage that G is compact. Now we turn to general locally compact groups. Here a complete classification seems hopeless, and even Abelianness fails in general (see Example 3.16 below). The following theorem is what we can prove:

Theorem 3.14. *Any thin crossed module of locally compact groups is equivalent to a thin crossed module $\mathcal{C}' = (G', H', \partial', c')$ with compact G' and discrete H'.*

The following are equivalent:

(1) *$\mathcal{C}$ is equivalent to a thin crossed module $\mathcal{C}'$ with compact Abelian G', discrete H', and trivial c';*

(2) *$\mathcal{C}$ is equivalent to a crossed module $\mathcal{C}'$ with trivial c';*
(3) *the commutator map $G\times G\to G$, $(x,y)\mapsto xyx^{-1}y^{-1}$, factors as $\partial\circ\gamma$ for a continuous map $\gamma\colon G\times G\to H$.*

Proof. We construct a finite chain of crossed modules $\mathcal{C}_i=(G_i,H_i,\partial_i,c_i)$ equivalent to $\mathcal{C}$ with increasingly better properties. Each step uses Example 3.7 or 3.8, which describe how the maps ∂_i and c_i and one of the groups in the next step are constructed. We must only describe the subgroup to which we restrict in Example 3.7 or the normal subgroup we divide out in Example 3.8.

By the structure theory of locally compact groups, the group G contains an open, almost connected subgroup G_1. Restricting to this subgroup as in Example 3.7, we get an equivalent crossed module $\mathcal{C}_1$ with almost connected G_1.

Next we divide out suitable compact, c_1-invariant (hence normal) subgroups in H_1. On such subgroups, the map ∂_1 is automatically a homeomorphism onto a compact subgroup of G_3, so that we may divide it out using Example 3.8. Let H_1^0 be the connected component of the identity in H_1. This closed subgroup is intrinsically defined and hence c_1-invariant. Being connected, it contains a compact normal subgroup N so that H_1^0/N is a Lie group. If N_1 and N_2 are such subgroups, then so is $N_1\cdot N_2$ because of normality. Hence there is a maximal compact normal subgroup N_1 in H_1^0. This subgroup of H_1^0 is intrinsically defined and hence c_1-invariant. We may now pass to an equivalent quotient crossed module $\mathcal{C}_2$ with $H_2=H_1/N_1$. The group G_2 is still almost connected, and $H_2^0=H_1^0/N_1$ is now a Lie group.

Since G_2 is almost connected, it contains a decreasing net of compact normal subgroups K_i such that G_2/K_i are Lie groups and $\bigcap K_i=\{1\}$. The map ∂_1 induces Lie algebra homomorphisms from the Lie algebra $\mathfrak{h}_2$ of H_2^0 to the Lie algebras of G_2/K_i. Since ∂_1 is injective and $\mathfrak{h}_2$ is finite-dimensional, this map is injective for some i. Thus there is a compact normal subgroup $K\subseteq G_2$ such that G_2/K is a Lie group and the map $\mathfrak{h}_2\to\mathfrak{g}_2/\mathfrak{k}$ is injective.

Passing to a subgroup and a covering group $G_3\to G_2$ as in the proof of Theorem 3.11, we now find an equivalent crossed module $\mathcal{C}_3$ for which G_3/K is simply connected. The Lie algebras of H_2 and H_3 are the same and the map $\mathfrak{h}_3\to\mathfrak{g}_3/\mathfrak{k}$ remains injective. Since G_3/K is simply connected, there is a unique connected, closed, normal subgroup L_3 of G_3/K whose Lie algebra is the image of $\mathfrak{h}_3$ (see the proof of Theorem 3.11); a long exact sequence of homotopy groups using that π_2 vanishes for all Lie groups (such as G_3/KL_3) shows that L_3 is simply connected. The map $H_3^0\to L_3$ is the identity on Lie algebras and hence a covering map. Since L_3 is already simply connected, it is a homeomorphism. Then the map $\partial_3\colon H_3^0\to G_3$ is a homeomorphism onto its image as well. Hence Example 3.8 gives us an equivalent crossed module $\mathcal{C}_4$ with $H_4=H_3/H_3^0$. Now H_4 is totally disconnected, and G_4 is still almost connected.

Since H_4 is totally disconnected, any action of a connected group on it is trivial. In particular, the conjugation action of G_4 factors through G_4/G_4^0. Since this group is compact, an intersection $U':=\bigcap_{g\in G_4}c_g(U)$ for an open subset $U\subseteq H_4$ is again open. If U is a compact open subgroup, then this intersection is a compact, open, and c_4-invariant subgroup in H_4. Example 3.8 gives us an equivalent crossed module $\mathcal{C}_5$ with $H_5=H_4/U'$. Now H_5 is discrete, and G_5 is still almost connected.

Let $g\in G_5^0$ belong to the connected component of the identity. Since G_5^0 is connected and H_5 discrete, $c_g(h)=h$ for all $h\in H_5$. Hence $g\partial_5(h)g^{-1}=\partial_5(h)$ for all $h\in H_5$. Since $\partial_5(H)$ is dense in G_5, this shows that g is central. Hence the connected component G_5^0 is a central subgroup in G_5.

The image of $K_3\subseteq G_3$ in G_5 is a compact normal subgroup K_5 such that G_5/K_5 is connected, so that G_5^0 surjects onto it. Since G_5^0 is central, the quotient group G_5/K_5

is a commutative Lie group. As in the proof of Theorem 3.11, we now find a subgroup $N_5 \subseteq H_5$ isomorphic to $\mathbb{Z}^l$ whose image in G_5/K_5 is a lattice. Dividing out this subgroup as in Example 3.8, we find an equivalent crossed module $\mathcal{C}_6$ where G_6 is compact and H_6 is still discrete. This proves our first statement.

Assume now that the commutator map in G factors through a continuous map $G \times G \to H$; since ∂ is injective, this just means that it is a map to H, and continuous as such. The crossed modules $\mathcal{C}_i$ constructed above inherit this property in each step. Thus the commutator map $\gamma\colon G_6 \times G_6 \to G_6$ is a continuous map to H_6. Since H_6 is discrete, there is an open subset $U \subseteq G_6$ with $\gamma(x, y) = 1$ for all $x, y \in U$, that is, $xy = yx$ for all $x, y \in U$. This remains so for x, y in the subgroup G' generated by U, which is open in G_6 because U is open. Example 3.7 shows that G' is part of a crossed module equivalent to $\mathcal{C}$. This has Abelian compact G', discrete H', and hence trivial c'.

As a consequence, 3.14.3 implies 3.14.1. It is trivial that 3.14.1 implies 3.14.2. It remains to show, that 3.14.3 follows, conversely, if $\mathcal{C}$ is equivalent to a crossed module $\mathcal{C}'$ with trivial c'. Since $\mathcal{C}'$ is again thin, this implies that G' and hence $\operatorname{coker} \partial' = \pi_1(\mathcal{C}')$ is Abelian. Then $\pi_1(\mathcal{C}) = G/\partial(H)$ is Abelian as well because π_1 is invariant under equivalence of crossed modules. This says that the commutator map of G factors through H; it does not yet give the continuity of the factorisation. We merely sketch how to prove this continuity. The commutator map of G' is constant and hence clearly a continuous map to H'. To finish the proof, we must show that the existence of a continuous commutator map $G \times G \to H$ for thin crossed modules is invariant under equivalence of crossed modules.

We turn the functor $M\colon (H \ltimes G) \times (H \ltimes G) \to (H \ltimes G)$ on the arrow groupoids in (3.4) into a generalised morphism (bispace); two functors give isomorphic bispaces if and only if they are related by conjugation with a bisection. For a thin crossed module, a bisection that conjugates M onto $M \circ \text{flip}$ is exactly the same as a continuous commutator map $G \times G \to H$. Thus the property of having a continuous commutator map $G \times G \to H$ is equivalent to the property that M and $M \circ \text{flip}$ are equivalent as generalised morphisms. This property is manifestly invariant under equivalence of crossed modules. □

Lemma 3.15. *Let $\mathcal{C} = (G, H, \partial, c)$ be a thin crossed module with a compact subset $K \subseteq H$ such that $\overline{\partial(K)}$ generates an open subgroup in G. Then $\mathcal{C}$ is equivalent to a crossed module $\mathcal{C}'$ with trivial c'.*

Proof. Call a subgroup of a locally compact group *compactly generated* if it is generated by a compact subset. Our assumption is that there is a compactly generated subgroup A of H for which $\overline{\partial(A)}$ is open in G. If $G_1 \subseteq G$ is an open subgroup, then $A_1 := \partial^{-1}(G_1) \cap A$ has the same property for the crossed module $\mathcal{C}_1$ constructed as in Example 3.7. If $\pi\colon G_2 \to G$ is a quotient mapping for which $\ker \pi$ is compactly generated, then $\mathcal{C}_2$ constructed as in Example 3.8 also inherits this property, taking the preimage of A in A_2, which is again finitely generated. The coverings needed in the proof of Theorem 3.14 have compactly generated kernels because we divide out either compact groups, connected Lie groups, or discrete subgroups in connected Lie groups, which are all compactly generated. Hence Theorem 3.14 provides an equivalent crossed module $\mathcal{C}'$ with compact G' and discrete H' and a compactly generated subgroup $A \subseteq H'$ for which $\overline{\partial(A)}$ is dense in G'. Since H' is discrete, A is generated by a finite subset S. Since c' is continuous, the set of $g \in G'$ with $gh = hg$ for all $h \in S$ is open in G'. Since S generates G' topologically, this open subgroup is central in G'. Thus G' has an open, finite-index centre Z. Now replace $\mathcal{C}'$ by an equivalent crossed module $\mathcal{C}''$ as in Example 3.7 with $G'' = Z$. This has commutative G'' and hence trivial c'' because ∂ is injective. □

Example 3.16. We construct an example of a thin crossed module of locally compact groups where $G/\partial(H)$ is not Abelian. Since $G/\partial(H) = \operatorname{coker}\partial$ is invariant under equivalence, this crossed module cannot be equivalent to a commutative one. Let G_1 be some finite group that is not Abelian. Let

$$H := \bigoplus_{n\in\mathbb{N}} G_1, \qquad G := \prod_{n\in\mathbb{N}} G_1,$$

where $\bigoplus G_1$ is the subgroup of all $(g_n) \in \prod G_1$ with $g_n = 1$ for all but finitely many entries. Hence H is a countable group; we give it the discrete topology. The group G is pro-finite and in particular compact. It contains H as a dense normal subgroup, and the conjugation map $g \mapsto ghg^{-1}$ for any fixed $h \in H$ factors through a finite product and therefore is continuous. The constant embedding $G_1 \to G$ remains an embedding into G/H. Hence G/H is not Abelian.

Some of our simplifications also work for crossed modules that are not thin. We give one such statement:

Proposition 3.17. *Let $\mathcal{C} = (G, H, \partial, c)$ be a crossed module of Lie groups with connected G and injective ∂. Then $\mathcal{C}$ is equivalent to a crossed module $\mathcal{C}' = (G', H', \partial', c')$ where G' is simply connected, H' is discrete, and c' is trivial. Two crossed modules of this form are equivalent if and only if they are isomorphic.*

Proof. As in the proof of Theorem 3.5, we may pass to an equivalent crossed module $\mathcal{C}_1$ where G_1 is the universal covering of G; in this new crossed module, the image of the connected component of H_1 is closed. Since ∂ is assumed injective, we may divide out this connected component and arrive at an equivalent crossed module with discrete H_2 and simply connected G_2. As in the proof of Theorem 3.5, it follows that the conjugation action on H_2 is trivial. Hence $\mathcal{C}_2$ has the asserted properties.

Now consider the invariant $(\mathfrak{g}/\mathfrak{h}, T(\mathcal{C}))$ under equivalence used already in the proof of Theorem 3.5. If $\mathcal{C}_1$ and $\mathcal{C}_2$ are two crossed modules with discrete H, simply connected G and injective ∂, then an isomorphism $\mathfrak{g}_1/\mathfrak{h}_1 \to \mathfrak{g}_2/\mathfrak{h}_2$ is an isomorphism $\mathfrak{g}_1 \to \mathfrak{g}_2$ because H_1 and H_2 are discrete. This lifts to an isomorphism $G_1 \to G_2$ because G_1 and G_2 are simply connected. Furthermore, we claim that the exponential maps $\mathfrak{g}_i \to G_i$ map $T(\mathcal{C}_i)$ onto $\partial(H_i)$. This is because $\partial(H_i)$ is contained in the centre of G_i, the centre of G_i is isomorphic to $\mathbb{R}^n$ for some n because G_i is simply connected, and the exponential map for $\mathbb{R}^n$ is surjective. Therefore, an isomorphism between the invariants $(\mathfrak{g}_1/\mathfrak{h}_1, T(\mathcal{C}_1))$ and $(\mathfrak{g}_2/\mathfrak{h}_2, T(\mathcal{C}_2))$ already implies that $\mathcal{C}_1$ and $\mathcal{C}_2$ are isomorphic crossed modules. □

4. Duality for Abelian crossed modules

Definition 4.1. We call a crossed module (G, H, ∂, c) 2-*Abelian* if the action c is trivial, and *Abelian* if c is trivial and G is Abelian.

For a 2-Abelian crossed module, H is Abelian because $hkh^{-1} = c_{\partial(h)}(k)$ for all $h, k \in H$, and $\partial(H)$ is a central subgroup of G by $\partial(c_g(h)) = g\partial(h)g^{-1}$. For a thin crossed module, H is Abelian if and only if c is trivial, if and only if G is Abelian. Hence, for thin crossed modules, Abelianness and 2-Abelianness are equivalent conditions. Of course, in general a 2-Abelian crossed module need not be Abelian (for instance, any group G viewed as a crossed module $(G, 0, 0, 0)$ is 2-Abelian). Proposition 3.17 implies that any crossed module of Lie groups with connected G is equivalent to a 2-Abelian one.

Let $\mathcal{C} = (G, H, \partial, c)$ be a 2-Abelian crossed module of locally compact groups. Let $(\mathcal{A}, \upsilon)$ be a Fell bundle over $\mathcal{C}$. Condition (2) in Definition 2.10 now says that $a \cdot \upsilon_h = \upsilon_h \cdot a$ for all $a \in \mathcal{A}$, $h \in H$. This extends to $a \in \mathrm{C_c}(\mathcal{A})$ and then

to $a \in \mathrm{C}^*(\mathcal{A})$; here we embed υ_h into the multiplier algebra of $\mathrm{C}^*(\mathcal{A})$ using the universal representation ρ^{u}. Thus ρ^{u} now maps H to the centre of $\mathcal{M}(\mathrm{C}^*(\mathcal{A}))$. This map integrates to a nondegenerate *-homomorphism

$$\textstyle\int\rho^{\mathrm{u}}\colon \mathrm{C}^*(H) \to Z\mathcal{M}(\mathrm{C}^*(\mathcal{A})). \tag{4.2}$$

Identifying $\mathrm{C}^*(H)$ with $\mathrm{C}_0(\hat{H})$ for the Pontryagin dual $\hat{H}$ of H, we get a structure of $\mathrm{C}_0(\hat{H})$-algebra on $\mathrm{C}^*(\mathcal{A})$. (We normalise the Fourier transform so that the unitary $\delta_h \in \mathcal{UM}(\mathrm{C}^*(H))$ becomes the function $\hat{h} \mapsto \hat{h}(h)$ on $\hat{H}$.)

Proposition 4.3. *The crossed product* $\mathrm{C}^*(\mathcal{A}, \upsilon)$ *is the fibre at* $1 \in \hat{H}$ *for the* $\mathrm{C}_0(\hat{H})$-C^**-algebra structure on* $\mathrm{C}^*(\mathcal{A})$ *just described.*

Proof. By construction, $\mathrm{C}^*(\mathcal{A}, \upsilon)$ is the quotient by the relation $\rho^{\mathrm{u}}(\upsilon_h) = 1$ for all $h \in H$. This is equivalent to $\int\rho^{\mathrm{u}}(f) = f(1_{\hat{H}})$ for all $f \in \mathrm{C}_0(\hat{H}) \cong \mathrm{C}^*(H)$. Hence $\mathrm{C}^*(\mathcal{A}, \upsilon)$ is the quotient by $\mathrm{C}_0(\hat{H} \setminus \{1\}) \cdot \mathrm{C}^*(\mathcal{A})$, that is, the fibre at 1. □

The above proposition is merely an observation, the main point is to see that $\mathrm{C}^*(\mathcal{A})$ has a relevant $\mathrm{C}_0(\hat{H})$-algebra structure. This proposition allows us to compute crossed products by 2-Abelian crossed modules in two more elementary steps: first take the crossed product $\mathrm{C}^*(\mathcal{A})$ by the locally compact group G; then take the fibre at $1 \in \hat{H}$ for the canonical $\mathrm{C}_0(\hat{H})$-algebra structure on $\mathrm{C}^*(\mathcal{A})$.

Example 4.4. Let $\theta \in \mathbb{R} \setminus \mathbb{Q}$ and let $\mathrm{C}^*(\mathbb{T}_\theta)$ be the associated noncommutative torus. It carries a strict action by the Abelian crossed module $\partial\colon \mathbb{Z} \to \mathbb{T}$ with $\partial(n) := \exp(2\pi\mathrm{i}\theta n)$; namely, $\mathbb{T}$ acts by part of the gauge action: $\alpha_z(U) := zU$, $\alpha_z(V) := V$, where U and V are the standard generators of $\mathrm{C}^*(\mathbb{T}_\theta)$, and $\mathbb{Z}$ acts by $n \mapsto V^{-n}$ (see [1]). The crossed product for this action is already computed in [1]: it is the C^*-algebra of compact operators on the Hilbert space $L^2(\mathbb{T})$. This also follows from Proposition 4.3. First, the crossed product $\mathrm{C}^*(\mathbb{T}_\theta) \rtimes \mathbb{T}$ turns out to be $\mathrm{C}(\mathbb{T}) \otimes \mathbb{K}(L^2\mathbb{T})$ because the $\mathbb{T}$-action on $\mathrm{C}^*(\mathbb{T}_\theta)$ is a dual action if we interpret $\mathrm{C}^*(\mathbb{T}_\theta)$ as $\mathrm{C}(\mathbb{T}) \rtimes \mathbb{Z}$ with $\mathrm{C}(\mathbb{T}) = \mathrm{C}^*(V^*)$ and U generating $\mathbb{Z}$. The map $\mathrm{C}^*(\mathbb{Z}) \to \mathrm{C}^*(\mathbb{T}_\theta) \rtimes \mathbb{T}$ becomes an isomorphism onto $\mathrm{C}(\mathbb{T}) \otimes 1$. Hence the fibre at $1 \in \hat{\mathbb{Z}} = \mathbb{T}$ gives $\mathbb{K}(L^2\mathbb{T})$ as expected.

Next we generalise Takesaki–Takai duality from Abelian groups to Abelian crossed modules. We first reformulate the classical statement in our setting of correspondence 2-categories.

Let G be an Abelian locally compact group and let $\hat{G}$ be its Pontryagin dual. A crossed product for a G-action carries a canonical dual action of $\hat{G}$ and a crossed product for a $\hat{G}$-action carries a canonical dual action of G. These constructions extend to map G-equivariant correspondences to $\hat{G}$-equivariant correspondences, and vice versa, such that equivariant isomorphism of correspondences is preserved. That is, they provide functors $\mathfrak{Corr}(G) \leftrightarrow \mathfrak{Corr}(\hat{G})$. Takesaki–Takai duality implies that these functors are equivalences of 2-categories inverse to each other. Going back and forth gives equivariant isomorphisms $A \rtimes G \rtimes \hat{G} \cong A \otimes \mathbb{K}(L^2G)$ and $A \rtimes \hat{G} \rtimes G \cong A \otimes \mathbb{K}(L^2\hat{G})$, respectively; we interpret these as equivariant Morita–Rieffel equivalences $A \rtimes \hat{G} \rtimes G \simeq A$ and $A \rtimes G \rtimes \hat{G} \simeq A$ via $A \otimes L^2G$ and $A \otimes L^2\hat{G}$, respectively. These equivariant Morita–Rieffel equivalences are natural with respect to equivariant correspondences, hence the functors we get by composing the crossed product functors are naturally equivalent to the identity functors.

Now let $\mathcal{C} = (G, H, \partial, 0)$ be an Abelian crossed module of locally compact groups; this is nothing but a continuous homomorphism between two locally compact Abelian groups. Its *dual crossed module* $\hat{\mathcal{C}}$ consists of the dual groups $\hat{G}$ and $\hat{H}$ and the transpose $\hat{\partial}\colon \hat{H} \to \hat{G}$ of ∂. The bidual of $\mathcal{C}$ is naturally isomorphic to $\mathcal{C}$

because this holds for locally compact groups. Our duality theorem does not compare the action 2-categories of the crossed modules $\mathcal{C}$ and $\hat{\mathcal{C}}$, however; instead, the action 2-category of the arrow groupoid $\hat{G} \ltimes \hat{H}$ of $\hat{\mathcal{C}}$ appears. And the functors going back and forth are the crossed product functors for the groups G and $\hat{G}$.

We now describe the equivalence of 2-categories $\mathfrak{Corr}(\mathcal{C}) \simeq \mathfrak{Corr}(\hat{G} \ltimes \hat{H})$. Let $(\mathcal{A}, \upsilon)$ be a saturated Fell bundle over $\mathcal{C}$, encoding a $\mathcal{C}$-action by correspondences. The cross-sectional C^*-algebra $\mathrm{C}^*(\mathcal{A})$ carries a dual action of $\hat{G}$, where a character acts by pointwise multiplication on sections. Since Abelian crossed modules are 2-Abelian, (4.2) provides a nondegenerate *-homomorphism

$$\textstyle\int \rho^{\mathrm{u}} \colon \mathrm{C}_0(\hat{H}) \cong \mathrm{C}^*(H) \to Z\mathcal{M}(\mathrm{C}^*\mathcal{A}).$$

Since υ_h is a multiplier of degree $\partial(h)$, the dual action α of $\hat{G}$ acts on $\delta_h \in \mathrm{C}^*(H)$ by $\alpha_{\hat{g}}(\delta_h) = \hat{g}(\partial h) \cdot \delta_h$. This defines an action of $\hat{G}$ on $\mathrm{C}^*(H)$, which corresponds to the action $\rho_{\hat{g}} f(\hat{h}) = f(\hat{h} \cdot \hat{\partial}(\hat{g}))$ on $\mathrm{C}_0(\hat{H})$ coming from the right translation action of $\hat{G}$ on $\hat{H}$ through the homomorphism $\hat{\partial} \colon \hat{G} \to \hat{H}$. Thus $\mathrm{C}^*(\mathcal{A})$ carries a strict, continuous action of the arrow groupoid $\hat{G} \ltimes \hat{H}$ of $\hat{\mathcal{C}}$.

The actions of $\hat{G} \ltimes \hat{H}$ form a 2-category $\mathfrak{Corr}(\hat{G} \ltimes \hat{H})$ with actions by correspondences as objects, equivariant correspondences as arrows and isomorphisms of equivariant correspondences as 2-arrows. We claim that the construction on the level of objects above is part of a functor $\mathfrak{Corr}(\mathcal{C}) \to \mathfrak{Corr}(\hat{G} \ltimes \hat{H})$.

Let $\mathcal{E}$ be a correspondence of Fell bundles $(\mathcal{A}, \upsilon^{\mathcal{A}}) \to (\mathcal{B}, \upsilon^{\mathcal{B}})$. The space $\mathrm{C}_c(\mathcal{E})$ of compactly supported, continuous sections of $\mathcal{E}$ is a pre-Hilbert module over the *-algebras $\mathrm{C}_c(\mathcal{A})$ and $\mathrm{C}_c(\mathcal{B})$, respectively. Using the map $\mathrm{C}_c(\mathcal{B}) \subseteq \mathrm{C}^*(\mathcal{B})$, we may complete $\mathrm{C}_c(\mathcal{E})$ to a Hilbert $\mathrm{C}^*(\mathcal{B})$-module $\mathrm{C}^*(\mathcal{E})$. The left action of $\mathrm{C}_c(\mathcal{A})$ on $\mathrm{C}_c(\mathcal{E})$ induces a left action of $\mathrm{C}^*(\mathcal{A})$ on $\mathrm{C}^*(\mathcal{E})$, turning it into a correspondence $\mathrm{C}^*(\mathcal{A}) \to \mathrm{C}^*(\mathcal{B})$. Isomorphic correspondences of Fell bundles clearly give isomorphic correspondences $\mathrm{C}^*(\mathcal{A}) \to \mathrm{C}^*(\mathcal{B})$. Thus taking cross-sectional C^*-algebras gives a functor $\mathfrak{Corr}(\mathcal{C}) \to \mathfrak{Corr}$ (where $\mathfrak{Corr} = \mathfrak{Corr}(\{1\})$ denotes the 2-category of C^*-algebras with correspondences as their morphisms); actually, since we did not use the H-action, this is just the crossed product functor $\mathfrak{Corr}(G) \to \mathfrak{Corr}$, composed with the forgetful functor $\mathfrak{Corr}(\mathcal{C}) \to \mathfrak{Corr}(G)$.

Pointwise multiplication by characters defines a $\hat{G}$-action on $\mathrm{C}_c(\mathcal{E})$. This extends to the completion $\mathrm{C}^*(\mathcal{E})$, and turns it into a $\hat{G}$-equivariant correspondence $\mathrm{C}^*(\mathcal{A}) \to \mathrm{C}^*(\mathcal{B})$. This $\hat{G}$-action is natural for isomorphisms of Fell bundle correspondences. Furthermore, $\mathrm{C}^*(\mathcal{E})$ is a $\mathrm{C}^*(H)$-linear correspondence because $\upsilon_h^{\mathcal{A}} \cdot \xi = \xi \cdot \upsilon_h^{\mathcal{B}}$ for all $h = c_g(h) \in H$ by 2.13.(7). Thus taking the cross-sectional C^*-algebra (that is, the crossed product by G) gives a functor $\mathfrak{Corr}(\mathcal{C}) \to \mathfrak{Corr}(\hat{G} \ltimes \hat{H})$.

Theorem 4.5. *Let $\mathcal{C}$ be an Abelian crossed module. The functor $\mathfrak{Corr}(\mathcal{C}) \to \mathfrak{Corr}(\hat{G} \ltimes \hat{H})$ just described is an equivalence of 2-categories.*

Proof. To prove the existence of a quasi-inverse functor $\mathfrak{Corr}(\hat{G} \ltimes \hat{H}) \to \mathfrak{Corr}(\mathcal{C})$, it suffices to construct it for strict actions by automorphisms because arbitrary actions by correspondences are equivalent to strict actions by automorphisms (see [2, Theorem 5.3]), where "equivalent" means "isomorphic in the 2-category $\mathfrak{Corr}(\hat{G} \ltimes \hat{H})$."

Thus let B be a C^*-algebra with a continuous action of $\hat{G} \ltimes \hat{H}$ in the usual sense. This consists of a strict action β of the group $\hat{G}$ and a $\hat{G}$-equivariant nondegenerate *-homomorphism from $\mathrm{C}_0(\hat{H})$ to $Z\mathcal{M}(B)$. The crossed product $B \rtimes \hat{G}$ carries a dual action $\hat{\beta}$ of G. For $h \in H$, define $v_h \in \mathcal{UM}(\mathrm{C}_0(\hat{H}))$ by $v_h(\hat{h}) = \hat{h}(h)^{-1}$ for all $\hat{h} \in \hat{H}$. The right translation action of $\hat{G}$ acts on v_h by $\hat{g}(v_h) = \hat{g}(\partial(h))^{-1} \cdot v_h$

because

$$\hat{g}(v_h)(\hat{h}) = v_h(\hat{h}\cdot\partial\hat{g}) = (\partial\hat{g})(h)^{-1}\hat{h}(h)^{-1} = \hat{g}(\partial h)^{-1}v_h(\hat{h})$$

for all $\hat{h}\in\hat{H}$. Now map v_h to $\mathcal{UM}(B\rtimes\hat{G})$ using the homomorphism $\mathcal{UM}(\mathrm{C}_0(\hat{H}))\to\mathcal{UM}(B\rtimes\hat{G})$ induced by the $\mathrm{C}_0(\hat{H})$-C^*-algebra structure on B. These unitaries $v_h\in\mathcal{UM}(B\rtimes\hat{G})$ commute with B because $\mathrm{C}_0(\hat{H})$ is mapped to the centre of B, and they satisfy

$$v_h\delta_{\hat{g}}v_h^* = \delta_{\hat{g}}\hat{g}(\partial h)\cdot v_hv_h^* = \hat{g}(\partial h)\cdot\delta_{\hat{g}}$$

for $\hat{g}\in\hat{G}$. Hence $\hat{\beta}_{\partial(h)} = \mathrm{Ad}(v_h)$ for all $h\in H$. Since $\hat{\beta}_g(v_h) = v_h$ for all $g\in G$, $h\in H$, the map $h\mapsto v_h$ and the dual action $\hat{\beta}$ of G combine to a strict action of the crossed module $\mathcal{C}$ on $B\rtimes\hat{G}$.

The above constructions extend to equivariant correspondences in a natural way and thus provide a functor $\mathfrak{Corr}(\hat{G}\ltimes\hat{H})\to\mathfrak{Corr}(\mathcal{C})$. We claim that this is quasi-inverse to the functor $\mathfrak{Corr}(\mathcal{C})\to\mathfrak{Corr}(\hat{G}\ltimes\hat{H})$ constructed above. We must compose these functors in either order and check that the resulting functors are equivalent to the identity functors. Since all actions by correspondences are equivalent to strict actions by automorphisms, it is enough to verify the equivalence on those objects in $\mathfrak{Corr}(\mathcal{C})$ and $\mathfrak{Corr}(\hat{G}\ltimes\hat{H})$ that are strict actions by automorphisms.

Let $\alpha\colon G\to\mathrm{Aut}(A)$ and $u\colon H\to\mathcal{UM}(A)$ be a strict action of $\mathcal{C}$ on A. Our functor to $\mathfrak{Corr}(\hat{G}\ltimes\hat{H})$ maps it to $A\rtimes G$ equipped with the dual action of $\hat{G}$ and with the canonical map $\mathrm{C}_0(\hat{H})\cong\mathrm{C}^*(H)\to Z\mathcal{M}(A\rtimes G)$ induced by the representation $h\mapsto u_h^*\delta_{\partial(h)}$ for $h\in H$; this is how the translation from strict actions to Fell bundles works on the level of the unitaries u_h and v_h (see Example 2.12). The map to actions of $\mathcal{C}$ takes this to $(A\rtimes G)\rtimes\hat{G}$ equipped with the dual action of G and the homomorphism $H\to\mathcal{UM}(A\rtimes G\rtimes\hat{G})$, $h\mapsto(u_h^*\delta_{\partial(h)})^{-1}\in\mathcal{UM}(A\rtimes G)\subseteq\mathcal{UM}(A\rtimes G\rtimes\hat{G})$ because the isomorphism $\mathrm{C}_0(\hat{H})\cong\mathrm{C}^*(H)$ maps $v_h\mapsto\delta_{h^{-1}}$.

Takesaki–Takai duality provides a canonical G-equivariant isomorphism

$$A\rtimes G\rtimes\hat{G}\cong A\otimes\mathbb{K}(L^2G)\cong\mathbb{K}(A\otimes L^2G).$$

It extends the standard representation of $A\rtimes G$ on $A\otimes L^2G$ that maps $a\in A$ to the operator $\varphi(a)$ of pointwise multiplication by $G\ni g\mapsto\alpha_g(a)$ and $g\in G$ to the right translation operator $\rho_gf(g') = f(g'g)$ for all $g'\in G$. Hence $(u_h^*\delta_{\partial(h)})^{-1} = \delta_{\partial(h)^{-1}}u_h$ acts by the unitary operator

$$(\rho_{\partial(h)^{-1}}\varphi(u_h)f)(g) = u_h\cdot f(g\partial(h)^{-1}) = u_h\cdot f(\partial(h)^{-1}g)$$

because G is Abelian and $\alpha_g(u_h) = u_h$ for all $g\in G$, $h\in H$. This gives the operator $u_h\otimes\lambda_{\partial(h)}$ for the left regular representation λ on G. The imprimitivity bimodule $A\otimes L^2G$ between $\mathbb{K}(A\otimes L^2G)\cong A\rtimes G\rtimes\hat{G}$ and A with the G-action $g\mapsto\alpha_g\otimes\lambda_g$ is $\mathcal{C}$-equivariant and thus provides a $\mathcal{C}$-equivariant equivalence $A\simeq A\rtimes G\rtimes\hat{G}$.

Now let B carry a strict $\hat{G}\ltimes\hat{H}$-action. That is, $\hat{G}$ acts on B via an action β, and we have a nondegenerate $\hat{G}$-equivariant *-homomorphism $\phi\colon\mathrm{C}_0(\hat{H})\to Z\mathcal{M}(B)$. Our functor to $\mathfrak{Corr}(\mathcal{C})$ takes this to $B\rtimes\hat{G}$ equipped with the dual action of G and the homomorphism $H\to\mathcal{U}Z\mathcal{M}(B\rtimes\hat{G})$ defined as the composition of the representation $v\colon H\to Z\mathcal{M}(B)$, $h\mapsto\phi(v_h)$ with $v_h(\hat{h}) := \hat{h}(h)^{-1}$ and the canonical embedding $B\to\mathcal{M}(B\rtimes\hat{G})$. The functor that goes back to a $\hat{G}\ltimes\hat{H}$-action now gives $B\rtimes\hat{G}\rtimes G$ with the dual action of $\hat{G}$ and with the nondegenerate *-homomorphism $\mathrm{C}^*(H)\to Z\mathcal{M}(B\rtimes\hat{G}\rtimes G)$ that maps $h\in H$ to $\phi(v_h^*)\delta_{\partial(h)}$ – where here $\phi(v_h)$ is viewed as an element of $\mathcal{M}(B\rtimes\hat{G}\rtimes G)$ using the canonical homomorphism $B\to\mathcal{M}(B\rtimes\hat{G}\rtimes G)$. Now we identify $\mathrm{C}_0(\hat{H})\cong\mathrm{C}^*(H)$ as before, mapping the character v_h^* to δ_h. Thus the resulting map $\mathrm{C}_0(\hat{H})\to Z\mathcal{M}(B\rtimes\hat{G}\rtimes G)$ maps v_h^* to $\phi(v_h^*)\delta_{\partial(h)}$. On the other hand, via the isomorphism $\mathrm{C}_0(\hat{H})\cong\mathrm{C}^*(H)$, the

homomorphism $\phi\colon \mathrm{C}_0(\hat{H}) \to \mathcal{M}(B)$ corresponds to the integrated form of the representation $H \to \mathcal{M}(B)$, $h \mapsto \phi(v_h^*)$. The $\hat{G}$-equivariance of ϕ means that $\beta_{\hat{g}}(\phi(v_h^*)) = \hat{g}(\partial(h)) \cdot \phi(v_h^*)$ (recall that the $\hat{G}$-action on $\mathrm{C}_0(\hat{H})$ is induced from the right translation $\hat{G}$-action on $\hat{H}$ via $(\hat{h}, \hat{g}) \mapsto \hat{h} \cdot \hat{\partial}(\hat{g})$).

Takesaki–Takai duality gives a $\hat{G}$-equivariant isomorphism

$$B \rtimes \hat{G} \rtimes G \cong B \otimes \mathbb{K}(L^2\hat{G}) \cong \mathbb{K}(B \otimes L^2\hat{G}).$$

It restricts to the standard representation of $B \rtimes \hat{G}$ on $B \otimes L^2\hat{G}$, where $b \in B$ acts by $b \cdot f(\hat{g}) = \beta_{\hat{g}}(b) f(\hat{g})$ and $\hat{g} \in \hat{G}$ acts by right translation; the representation of G lets $g \in G$ act by $g \cdot f(\hat{g}) = \hat{g}(g)^{-1} f(\hat{g})$ for all $f \in \mathrm{C_c}(\hat{G}, B) \subseteq B \otimes L^2\hat{G}$. Since $\beta_{\hat{g}}(\phi(v_h^*)) = \hat{g}(\partial(h)) \cdot \phi(v_h^*)$, we get

$$(\phi(v_h^*)\delta_{\partial(h)}) \cdot f(\hat{g}) = \hat{g}(\partial(h)) \cdot \phi(v_h^*) \cdot \hat{g}(\partial(h))^{-1} \cdot f(\hat{g}) = \phi(v_h^*) \cdot f(\hat{g}).$$

This means that the $\hat{G}$-equivariant $B \rtimes \hat{G} \rtimes G$-$B$-equivalence bimodule $B \otimes L^2\hat{G}$ also intertwines the representations of H (and hence and the representations of $\mathrm{C}_0(\hat{H})$) on $B \rtimes \hat{G} \rtimes G$ and B. In other words, $B \otimes L^2\hat{G}$ is a $\hat{G} \ltimes \hat{H}$-equivariant Morita–Rieffel equivalence between B and $B \rtimes \hat{G} \rtimes G$. □

Despite Theorem 4.5, there is an important difference between $\mathfrak{Corr}(\mathcal{C})$ and $\mathfrak{Corr}(\hat{G} \ltimes \hat{H})$. Both 2-categories come with a natural tensor product structure: take the diagonal action on the tensor product in $\mathfrak{Corr}(\mathcal{C})$, or the diagonal action on the tensor product over the base space $\hat{H}$ in $\mathfrak{Corr}(\hat{G} \ltimes \hat{H})$. These tensor products are quite different. In terms of $\mathfrak{Corr}(\hat{G} \ltimes \hat{H})$, the natural tensor product in $\mathfrak{Corr}(\mathcal{C})$ does the following. Take two C^*-algebras A_1 and A_2 with actions of $\hat{G} \ltimes \hat{H}$. Their tensor product is a C^*-algebra over $\hat{H} \times \hat{H}$ with a compatible action of the group $\hat{G} \times \hat{G}$. Restrict the group action to the diagonal (this gives the usual diagonal action). But instead of restricting the C^*-algebra to the diagonal in $\hat{H} \times \hat{H}$ as usual, give it a structure of $\mathrm{C}_0(\hat{H})$-C^*-algebra using the comultiplication $\mathrm{C}_0(\hat{H}) \to \mathrm{C}_0(\hat{H} \times \hat{H})$.

Remark 4.6. There is a more symmetric form of our duality where both partners in the duality are of the same form: both are length-two chain complexes of locally compact Abelian groups

$$H \xrightarrow{d} G \xrightarrow{d} K,$$

with the dual of the form $\hat{K} \to \hat{G} \to \hat{H}$. An action of such a complex consists of an action of the crossed module $H \to G$ and an action of the groupoid $G \ltimes K$, where both actions contain the same action of the group G; here the groupoid $G \ltimes K$ is the transformation groupoid for the translation action $k \cdot g := k \cdot d(g)$ of G on K. Actually, these actions are the actions of the 2-groupoid with object space K, arrows $g\colon k \to k \cdot d(g)$, and 2-arrows $H\colon g \Rightarrow g \cdot d(h)$; the chain complex condition $d^2 = 0$ ensures that this is a strict 2-groupoid. Our proof above shows that the 2-groupoids $H \to G \to K$ and $\hat{K} \to \hat{G} \to \hat{H}$ have equivalent action 2-categories on C^*-algebras. Namely, an action of the crossed module from $H \to G$ on A induces an action of the groupoid rom $\hat{G} \to \hat{H}$ on $A \rtimes G$, and an action of the groupoid from $G \to K$ induces an action of the crossed module from $\hat{K} \to \hat{G}$ on $A \rtimes G$. Since both constructions involve the same dual action of $\hat{G}$, we get an action of $\hat{K} \to \hat{G} \to \hat{H}$ on $A \rtimes G$. Now the functor backwards has exactly the same form, and going back and forth is still a stabilisation functor. Since the stabilisation is compatible with the crossed module and groupoid parts of our actions, it is compatible with their combination to a 2-groupoid action.

In Section 3 we have studied equivalence of general crossed modules via homomorphisms. We end this section by a criterion for equivalences between Abelian crossed modules.

Proposition 4.7. *Let $\mathcal{C}_i = (G_i, H_i, \partial_i, 0)$, $i = 1, 2$, be Abelian crossed modules and let $(\varphi, \psi)\colon \mathcal{C}_1 \to \mathcal{C}_2$ a homomorphism. The following are equivalent:*

(1) *the homomorphism (φ, ψ) is an equivalence;*
(2) *the diagram*
$$H_1 \xrightarrow{\iota_1} G_1 \times H_2 \xrightarrow{\pi_2} G_2$$
is an extension of locally compact Abelian groups, where
$$\iota_1 := (\partial_1^{-1}, \psi)\colon H_1 \to G_1 \times H_2, \qquad h_1 \mapsto (\partial_1(h_1)^{-1}, \psi(h_1)),$$
$$\pi_2\colon G_1 \times H_2 \to G_2, \qquad (g_1, h_2) \mapsto \varphi(g_1)\partial_2(h_2);$$
(3) *the dual diagram*
$$\widehat{H_1} \xleftarrow{\widehat{\iota_1}} \widehat{G_1} \times \widehat{H_2} \xleftarrow{\widehat{\pi_2}} \widehat{G_2}$$
is an extension of locally compact Abelian groups;
(4) *the dual homomorphism $(\hat{\psi}, \hat{\varphi})\colon \hat{\mathcal{C}}_2 \to \hat{\mathcal{C}}_1$ is an equivalence.*

Proof. We shall use Lemma 3.3. Since inversion on a topological group is a homeomorphism, the map in condition (1) in Lemma 3.3 is a homeomorphism if and only if ι_1 is a homeomorphism onto $\ker \pi_2$. The map π_2 is the same one appearing in condition (2) in Lemma 3.3. This is a homomorphism since all groups involved are Abelian, and its kernel is exactly the image of ι_1. Therefore, the assumptions in Lemma 3.3 hold if and only if $H_1 \to G_1 \times H_2 \to G_2$ is a topological group extension. Since taking duals preserves this property, this is equivalent to $\widehat{H_1} \leftarrow \widehat{G_1} \times \widehat{H_2} \leftarrow \widehat{G_2}$ being a topological group extension. As above, this is equivalent to $(\hat{\psi}, \hat{\varphi})$ being an equivalence. □

5. Partial crossed products

Now we come to the factorisation of the crossed product functor for an extension of crossed modules. Let $\mathcal{C}_i = (G_i, H_i, \partial_i, c_i)$ for $i = 1, 2, 3$ be crossed modules of locally compact groups.

Definition 5.1. A diagram $\mathcal{C}_1 \to \mathcal{C}_2 \to \mathcal{C}_3$ of homomorphisms of crossed modules is called a *strict extension* of crossed modules if the resulting diagrams
$$H_1 \xrightarrow{\psi_1} H_2 \xrightarrow{\psi_2} H_3 \quad \text{and} \quad G_1 \xrightarrow{\varphi_1} G_2 \xrightarrow{\varphi_2} G_3$$
are extensions of locally compact groups. That is, ψ_1 is a homeomorphism onto the kernel of ψ_2 and ψ_2 is an open surjection, and similarly for φ_1 and φ_2.

Theorem 5.2. *Let $\mathcal{C}_1 \rightarrowtail \mathcal{C}_2 \twoheadrightarrow \mathcal{C}_3$ be a strict extension of crossed modules and let A be a* C*-*algebra with an action of $\mathcal{C}_2$ by correspondences. Then $A \rtimes \mathcal{C}_1$ carries a canonical action of $\mathcal{C}_3$ by correspondences such that $(A \rtimes \mathcal{C}_1) \rtimes \mathcal{C}_3$ is naturally isomorphic to $A \rtimes \mathcal{C}_2$.*

Proof. Since strict actions by automorphisms are notationally simpler, we first prove the result in case A carries a strict action by automorphisms. Then we reduce the general case to this special case. A strict action by automorphisms is given by group homomorphisms $\alpha\colon G_2 \to \mathrm{Aut}(A)$ and $u\colon H_2 \to \mathcal{UM}(A)$ that satisfy $\alpha_{\partial_2(h)} = \mathrm{Ad}_{u_h}$ for all $h \in H_2$ and $\alpha_g(u_h) = u_{c_g(h)}$ for all $g \in G_2$, $h \in H_2$. To simplify notation, we also view G_1 and H_1 as subgroups of G_2 and H_2, respectively, so that we drop the maps φ_1 and ψ_1.

The crossed product $A \rtimes \mathcal{C}_1$ is a quotient of the crossed product $A \rtimes G_1$ for the group G_1 by the ideal generated by the relation $u_h \sim \delta_{\partial_1(h)}$ for all $h \in H_1$; that is, we divide $A \rtimes G_1$ by the closed linear span of the subset $\{x \cdot (u_h - \delta_{\partial_1(h)}) \cdot y : h \in H_1, x, y \in A \rtimes G_1\}$.

A canonical action γ' of G_2 on $A \rtimes G_1$ is defined on the dense subalgebra $\mathrm{C_c}(G_1, A)$ by

$$(\gamma'_{g_2} f)(g_1) := \alpha_{g_2}(f(g_2^{-1} g_1 g_2))$$

for $g_2 \in G_2$, $g_1 \in G_1$, $f \in \mathrm{C_c}(G_1, A)$; this extends to the C^*-completion. The action c_2 of G_2 on H_2 leaves $H_1 \subseteq H_2 = \ker \psi_2$ invariant because ψ_2 is G_2-equivariant. Since γ'_g for $g \in G_2$ maps $u_h - \delta_{\partial_1(h)}$ to $u_{c_g(h)} - \delta_{\partial_1(c_g(h))}$ and $c_{G_2}(H_1) \subseteq H_1$, the action γ descends to an action of G_2 on $A \rtimes \mathcal{C}_1$.

Let U_g for $g \in G_1$ be the image of $\delta_g \in \mathcal{UM}(A \rtimes G_1)$ in $\mathcal{UM}(A \rtimes \mathcal{C}_1)$; this defines a homomorphism $G_1 \to \mathcal{UM}(A \rtimes \mathcal{C}_1)$ with $\gamma_g = \mathrm{Ad}(U_g)$ all $g \in G_1$. Also let $U_h \in \mathcal{UM}(A \rtimes \mathcal{C}_1)$ for $h \in H_2$ be the image of $u_h \in \mathcal{UM}(A)$ under the canonical map $A \to \mathcal{M}(A \rtimes G_1) \to \mathcal{M}(A \rtimes \mathcal{C}_1)$. We claim that $\gamma_{\partial_2(h)} = \mathrm{Ad}(U_h)$ for all $h \in H_2$. To see this, we notice first that $c_{2,g}(h)h^{-1} \in H_1 = \ker \psi_2$ for $g \in G_1$ because $\varphi_2(g) = 1$ implies $\psi_2(c_{2,g}(h)h^{-1}) = \psi_2(c_{3,\varphi_2(g)}(h)h^{-1}) = 1$. Hence $U_{c_{2,g^{-1}}(h)h^{-1}} = \delta_{\partial_1(c_{2,g^{-1}}(h)h^{-1})}$ holds in $\mathcal{UM}(A \rtimes \mathcal{C}_1)$. This implies

$$\begin{aligned} U_h \delta_g U_h^* &= \delta_g U_{c_{2,g^{-1}}(h)} U_{h^{-1}} = \delta_g U_{c_{2,g^{-1}}(h)h^{-1}} \\ &= \delta_g \delta_{\partial_1(c_{2,g^{-1}}(h)h^{-1})} = \delta_{g \partial_2(c_{2,g^{-1}}(h)) \partial_2(h)^{-1}} = \delta_{\partial_2(h) g \partial_2(h)^{-1}}. \end{aligned}$$

Since we have assumed $u_h a u_h^* = \alpha_{\partial_2(h)}(a)$ for all $a \in A$, we get $U_h x U_h^* = \gamma_{\partial_2(h)}(x)$ for all $x \in A \rtimes \mathcal{C}_1$.

If $g \in G_1$, $h \in H_2$, then $U_g U_h U_g^*$ is the image of $\alpha_g(u_h) = u_{c_{2,g}(h)}$, that is, $U_g U_h U_g^* = U_{c_{2,g}(h)}$. Thus the map $(g, h) \mapsto U_g U_h$ is a homomorphism $G_1 \ltimes H_2 \to \mathcal{UM}(A \rtimes \mathcal{C}_1)$ where the semidirect product uses the action $G_1 \subseteq G_2 \to \mathrm{Aut}(H_2)$ given by restricting c_2. Since $U_h = U_{\partial_1(h)}$ for $h_1 \in H_1 \subseteq H_2$, we get a homomorphism on $H := G_1 \ltimes H_2/\Delta(H_1)$ with the embedding $\Delta\colon H_1 \to G_1 \ltimes H_2$, $h \mapsto (\partial(h)^*, h)$. The map $G_1 \ltimes H_2 \to G_2$, $(g, h) \mapsto g \cdot \partial_2(h)$, is a group homomorphism which vanishes on $\Delta(H_1)$ and hence descends to a group homomorphism $\partial\colon H \to G_2$. We define a homomorphism $c'\colon G_2 \to \mathrm{Aut}(G_1 \ltimes H_2)$ by $c'_{g_2}(g_1, h_2) := (g_2 g_1 g_2^{-1}, c_{2,g_2}(h_2))$. This leaves $\Delta(H_1)$ invariant and hence descends to a homomorphism $c\colon G_2 \to \mathrm{Aut}(H)$. Putting all this together we get a crossed module $\mathcal{C} := (G_2, H, \partial, c)$ which acts on $A \rtimes \mathcal{C}_1$ by $\gamma\colon G_2 \to \mathrm{Aut}(A \rtimes \mathcal{C}_1)$ and $U\colon H \to \mathcal{UM}(A \rtimes \mathcal{C}_1)$.

The homomorphism ∂ maps $G_1 \subseteq H$ homeomorphically onto the closed normal subgroup $G_1 \subseteq G_2$. We have $G_2/\partial(G_1) \cong G_3$ and $H/G_1 \cong H_2/H_1 \cong H_3$. Example 3.8 shows that $\mathcal{C}$ is equivalent to the crossed module $\mathcal{C}_3$. By Theorem 3.5, the action (γ, U) of $\mathcal{C}$ on $A \rtimes \mathcal{C}_1$ is equivalent to an action of $\mathcal{C}_3$ on $A \rtimes \mathcal{C}_1$ by correspondences such that $(A \rtimes \mathcal{C}_1) \rtimes \mathcal{C} \cong (A \rtimes \mathcal{C}_1) \rtimes \mathcal{C}_3$. We claim that $(A \rtimes \mathcal{C}_1) \rtimes \mathcal{C} \cong A \rtimes \mathcal{C}_2$.

By the universal property a morphism $(A \rtimes \mathcal{C}_1) \rtimes \mathcal{C} \to D$ is equivalent to a $\mathcal{C}$-covariant representation of $A \rtimes \mathcal{C}_1$ in $\mathcal{M}(D)$, that is, a morphism $\rho_C\colon A \rtimes \mathcal{C}_1 \to \mathcal{M}(D)$ and a continuous homomorphism $V\colon G_2 \to \mathcal{UM}(D)$ with $V_g \rho_C(c) V_g^* = \rho_C(\gamma_g(c))$ for all $c \in A \rtimes \mathcal{C}_1$, $g \in G_2$ and $V_{\partial(h)} = \rho_C(U_h)$ for all $h \in H$. By the universal property of $A \rtimes \mathcal{C}_1$, the representation ρ_C is equivalent to a morphism $\rho_A\colon A \to \mathcal{UM}(D)$ and a continuous homomorphism $W\colon G_1 \to \mathcal{UM}(D)$ with $W_g \rho_A(a) W_g^* = \rho_A(\alpha_g(a))$ for all $a \in A$, $g \in G_1$ and $W_{\partial_1(h)} = \rho_A(u_h)$ for all $h \in H_1$. The assumptions on ρ_C are equivalent to $V_g \rho_A(a) V_g^* = \rho_A(\alpha_g(a))$ for $a \in A$, $g \in G_2$; $V_{g_2} W_{g_1} V_{g_2}^* = W_{g_2 g_1 g_2^{-1}}$ for $g_2 \in G_2$, $g_1 \in G_1$; $V_g = W_g$ for $g \in G_1$; and $V_{\partial_2(h)} = \rho_A(u_h)$ for $h \in H_2$. Thus the unitaries W_g for $g \in G_1$ are redundant, and the conditions on ρ_A and the unitaries V_g for $g \in G_2$, are precisely those for a covariant representation of A and $\mathcal{C}_2$. Hence the morphisms $(A \rtimes \mathcal{C}_1) \rtimes \mathcal{C} \to \mathcal{M}(D)$ are in natural bijection with morphisms $A \rtimes \mathcal{C}_2 \to \mathcal{M}(D)$. This shows that $A \rtimes \mathcal{C}_2 \cong (A \rtimes \mathcal{C}_1) \rtimes \mathcal{C}$. Since $(A \rtimes \mathcal{C}_1) \rtimes \mathcal{C} \cong (A \rtimes \mathcal{C}_1) \rtimes \mathcal{C}_3$, this gives the desired isomorphism.

We must show that the $\mathcal{C}_3$-action on $A \rtimes \mathcal{C}_1$ is natural, so that $A \mapsto (A \rtimes \mathcal{C}_1) \rtimes \mathcal{C}_3$ is a functor. Here we still talk about functors defined on the full sub-2-category of $\mathfrak{Corr}(\mathcal{C}_2)$ consisting of strict actions by automorphisms. The naturality of the $\mathcal{C}_3$-action is equivalent to the naturality of the $\mathcal{C}$-action by Theorem 3.5, which is what we are going to prove.

A $\mathcal{C}_2$-transformation between two strict actions on A_1 and A_2 by automorphisms is equivalent to a G-equivariant correspondence $\mathcal{E}$ from A_1 to A_2 in the usual sense, subject to the extra requirement $u_h^{A_1} \cdot \xi = \xi \cdot u_h^{A_2}$ for all $h \in H_2$, $\xi \in \mathcal{E}$. For such a correspondence, we get an action of G_2 on the induced correspondence $\mathcal{E} \rtimes \mathcal{C}_1$ from $A_1 \rtimes \mathcal{C}_1$ to $A_2 \rtimes \mathcal{C}_2$ by the same formulas as above, and this yields a $\mathcal{C}$-equivariant correspondence from $A_1 \rtimes \mathcal{C}_1$ to $A_2 \rtimes \mathcal{C}_2$. Furthermore, isomorphic $\mathcal{C}_2$-equivariant correspondences induce isomorphic $\mathcal{C}$-equivariant correspondences. Hence the $\mathcal{C}$-action on $A \rtimes \mathcal{C}_1$ is natural on the 2-category of strict actions of $\mathcal{C}_2$ by automorphisms.

The isomorphism $A \rtimes \mathcal{C}_2 \to (A \rtimes \mathcal{C}_1) \rtimes \mathcal{C}$ is natural in the sense that for any $\mathcal{C}_2$-equivariant correspondence $\mathcal{E}$ from A_1 to A_2, the square formed by the isomorphisms above and the induced correspondences $A_1 \rtimes \mathcal{C}_2 \to A_2 \rtimes \mathcal{C}_2$ and $(A_1 \rtimes \mathcal{C}_1) \rtimes \mathcal{C}_3 \to (A_2 \rtimes \mathcal{C}_1) \rtimes \mathcal{C}_3$ commutes up to a canonical isomorphism of correspondences. This establishes the naturality of our isomorphism on the 2-category of strict actions of $\mathcal{C}_2$.

By the Packer–Raeburn Stabilisation Trick, any action of $\mathcal{C}_2$ by correspondences is equivalent to a strict $\mathcal{C}_2$-action by automorphisms ([2, Theorem 5.3]), where equivalence means an isomorphism (that is, equivariant Morita equivalence) in the 2-category $\mathfrak{Corr}(\mathcal{C}_2)$. This equivalence means that a functor defined only on the subcategory of strict $\mathcal{C}_2$-actions may be extended to a functor on all of $\mathfrak{Corr}(\mathcal{C}_2)$; all such extensions are naturally isomorphic; and a natural transformation between functors on the subcategory extends to a natural transformation between the extensions. Hence the result for strict actions proves the more general result for actions by correspondences by abstract nonsense. □

Example 5.3. Let G be a locally compact group and let N be a closed normal subgroup of G so that we get a group extension $N \rightarrowtail G \twoheadrightarrow G/N$. Viewing N, G and G/N as crossed modules $\mathcal{C}_1 = (N, 0, 0, 0)$, $\mathcal{C}_2 = (G, 0, 0, 0)$ and $\mathcal{C}_3 = (G/N, 0, 0, 0)$, respectively, our result says that given an action α of G on a C^*-algebra A by correspondences, there is an action β of G/N on $A \rtimes_{\alpha|} N$ by correspondences, where $\alpha|$ denotes the restriction of α to N, such that

$$A \rtimes_\alpha G \cong (A \rtimes_{\alpha|} N) \rtimes_\beta G/N.$$

We may also interpret everything in terms of Fell bundles: the action of G on A corresponds to a Fell bundle $\mathcal{A}$ over G with unit fibre $\mathcal{A}_1 = A$ in such way that $A \rtimes_\alpha G$ is (isomorphic to) the cross-sectional C^*-algebra $\mathrm{C}^*(\mathcal{A})$. The restricted crossed product $A \rtimes_{\alpha|} N$ corresponds to the cross-sectional C^*-algebra $\mathrm{C}^*(\mathcal{A}_N)$ of the restriction $\mathcal{A}_N$ of $\mathcal{A}$ to N. Our theorem says that there is a Fell bundle $\mathcal{B}$ over G/N with unit fibre $\mathcal{B}_1 = A \rtimes_{\alpha|} N \cong C^*(\mathcal{A}|_N)$ such that $C^*(\mathcal{B}) \cong C^*(\mathcal{A})$. Although our C^*-algebraic version appears to be new, a version for L^1-cross-sectional algebras in proved by Doran and Fell in [3, VIII.6].

Even if we start with a strict action of G on A by *automorphisms*, the induced action of G/N on $A \rtimes N$ will usually not be an action by automorphisms. It may be interpreted as a Green twisted action of (G, N) on $A \rtimes N$, and the above decomposition corresponds to Green's decomposition of crossed products: $A \rtimes_\alpha G \cong (A \rtimes N) \rtimes (G, N)$ (see [4, 7]).

We may weaken the notion of strict extension by replacing the crossed modules involved by equivalent ones. We mention only one relevant example of this.

Example 5.4. Let $\mathcal{C} = (G, H, \partial, c)$ be a crossed module. Let $G_2 := G \ltimes_c H$ be the semidirect product group. It contains $H_2 := H$ as a normal subgroup via $\partial_2 \colon H_2 \to G_2$, $h \mapsto (1, h)$, with quotient $G_2/H_2 \cong G$. Let $c_2 \colon G_2 \to \operatorname{Aut}(H_2)$ be the resulting conjugation action, $c_{2,(g,h)}(k) := c_g(hkh^{-1})$. Then $\mathcal{C}_2 = (G_2, H_2, \partial_2, c_2)$ is a crossed module of locally compact groups that is equivalent to $(G, 0, 0, 0)$. Hence actions of $\mathcal{C}_2$ are equivalent to actions of the group G, with the same crossed products on both sides (Theorem 3.5). Let $\mathcal{C}_1 = (H, 0, 0, 0)$ be the group H turned into a crossed module and let $\mathcal{C}_3 = \mathcal{C}$. We map $\mathcal{C}_2 \to \mathcal{C}$ by $\psi_2 = \mathrm{Id} \colon H \to H$ and $\varphi_2 \colon G \ltimes_c H \to H$, $(g, h) \mapsto g \cdot \partial(h)$. This is a homomorphism of crossed modules and ψ_2 and φ_2 are open surjections. Their kernels are isomorphic to $H_1 := 0$ and $G_1 := H$ via $\varphi_1 \colon H \to G \ltimes_c H$, $h \mapsto (\partial(h)^{-1}, h)$, respectively. Thus we get a strict extension of crossed modules $\mathcal{C}_1 \to \mathcal{C}_2 \to \mathcal{C}$ with $\mathcal{C}_1 = (H, 0, 0, 0)$ and $\mathcal{C}_2 \simeq (G, 0, 0, 0)$. Hence the group G is equivalent to an extension of the group H by the crossed module $\mathcal{C}$.

Now let A carry an action of G, which we turn into an action of $\mathcal{C}_2$ via $G \ltimes H \to G$, $(g, h) \mapsto g\partial(h)$. When we apply Theorem 5.2 to this situation, we get back [1, Theorem 1]:

$$A \rtimes G \cong (A \rtimes H) \rtimes \mathcal{C}.$$

Example 5.5. Let θ be some irrational number and define an embedding $\theta \colon \mathbb{Z} \to \mathbb{R}$ by $n \mapsto \theta n$. Let $\mathcal{C} = (\mathbb{R}, \mathbb{Z}, \theta, 0)$ be the resulting Abelian crossed module. This is equivalent to the group $\mathbb{T}$ (viewed as the crossed module $(\mathbb{T}, 0, 0, 0)$) via the homomorphism $(\varphi, \psi) \colon \mathcal{C} \to \mathbb{T}$ with $\varphi \colon \mathbb{R} \to \mathbb{T}$, $t \mapsto \exp(2\pi\mathrm{i}\theta t)$, and the trivial homomorphism $\psi \colon \mathbb{Z} \to 0$ (Example 3.9). Theorem 3.5 gives an equivalence of 2-categories $\mathfrak{Corr}(\mathbb{T}) \xrightarrow{\sim} \mathfrak{Corr}(\mathcal{C})$; it sends a $\mathbb{T}$-algebra A to itself with $\mathbb{R}$ acting via φ and the given $\mathbb{T}$-action and $\mathbb{Z}$ acts trivially.

Now let $\mathcal{C}' = (\mathbb{T}, \mathbb{Z}, \partial, 0)$ be the crossed module considered in Example 4.4, where $\partial(n) = \exp(2\pi\mathrm{i}\theta n)$. View the group $\mathbb{Z}$ as a crossed module. There is an extension $\mathbb{Z} \rightarrowtail \mathcal{C} \twoheadrightarrow \mathcal{C}'$ described by the diagram:

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & \mathbb{Z} \arrow[r, "\mathrm{Id}"] \arrow[d, "\theta"'] & \mathbb{Z} \arrow[d, "\partial"] \\
\mathbb{Z} \arrow[r, "\mathrm{Id}"] & \mathbb{R} \arrow[r, "\varphi"] & \mathbb{T}
\end{tikzcd}$$

Therefore, Theorem 5.2 gives a functor $\mathfrak{Corr}(\mathcal{C}) \to \mathfrak{Corr}(\mathcal{C}')$ that sends a $\mathcal{C}$-algebra A to the (restricted) crossed product $A \rtimes \mathbb{Z}$ with an induced $\mathcal{C}'$-action, such that $A \rtimes \mathcal{C} \cong (A \rtimes \mathbb{Z}) \rtimes \mathcal{C}'$.

Composing this with the equivalence $\mathfrak{Corr}(\mathbb{T}) \cong \mathfrak{Corr}(\mathcal{C})$ we obtain a functor $\mathfrak{Corr}(\mathbb{T}) \to \mathfrak{Corr}(\mathcal{C}')$ that sends a $\mathbb{T}$-algebra A to the $\mathcal{C}'$-algebra $A \rtimes \mathbb{Z}$ with $(A \rtimes \mathbb{Z}) \rtimes \mathcal{C}' \cong A \rtimes \mathbb{T}$. As a simple example, we take the $\mathbb{T}$-algebra $\mathrm{C}(\mathbb{T})$ with translation $\mathbb{T}$-action. In this case, $\mathbb{Z}$ acts by irrational rotation by multiples of θ so that $\mathrm{C}(\mathbb{T}) \rtimes \mathbb{Z} \cong C^*(\mathbb{T}_\theta)$ is the noncommutative torus and the induced $\mathcal{C}'$-action is the same one considered in Example 4.4. Hence we get once again that $C^*(\mathbb{T}_\theta) \rtimes \mathcal{C}' \cong \mathrm{C}(\mathbb{T}) \rtimes \mathbb{T} \cong \mathbb{K}(L^2\mathbb{T})$.

Theorem 4.5 shows that $\mathfrak{Corr}(\mathcal{C}) \cong \mathfrak{Corr}(\hat{\mathbb{R}} \ltimes \hat{\mathbb{Z}}) \cong \mathfrak{Corr}(\mathbb{R} \ltimes \mathbb{T})$, where $\mathbb{R} \ltimes \mathbb{T}$ denotes the transformation groupoid for the action $t \cdot z = \exp(2\pi\mathrm{i}\theta t)z$ for $t \in \mathbb{R}$ and $z \in \mathbb{T}$. Composing $\mathfrak{Corr}(\mathbb{T}) \xrightarrow{\sim} \mathfrak{Corr}(\mathcal{C})$ with this equivalence, we get a functor $\mathfrak{Corr}(\mathbb{T}) \xrightarrow{\sim} \mathfrak{Corr}(\mathbb{R} \ltimes \mathbb{T})$. This takes a $\mathbb{T}$-algebra, views it as a $\mathcal{C}$-algebra, and sends it to the crossed product $A \rtimes \mathbb{R}$ viewed as an $\mathbb{R} \ltimes \mathbb{T}$-algebra using the dual $\mathbb{R}$-action and the structure of $\mathrm{C}(\mathbb{T})$-algebra given by the homomorphism $\mathrm{C}(\mathbb{T}) \cong \mathrm{C}^*(\mathbb{Z}) \to \mathrm{C}^*(\mathbb{R}) \to \mathcal{M}(A \rtimes \mathbb{R})$, which maps $\mathrm{C}(\mathbb{T})$ into the centre of $\mathcal{M}(A)$.

Theorem 4.5 also gives $\mathfrak{Corr}(\mathcal{C}') \cong \mathfrak{Corr}(\hat{\mathbb{T}} \ltimes \hat{\mathbb{Z}}) \cong \mathfrak{Corr}(\mathbb{Z} \ltimes \mathbb{T})$, where $\mathbb{Z}$ acts by rotation by multiples of θ. The quotient map $\mathcal{C} \to \mathcal{C}'$ becomes the forgetful functor that restricts an $\mathbb{R} \ltimes \mathbb{T}$-action to a $\mathbb{Z} \ltimes \mathbb{T}$-action on $\theta\mathbb{Z} \subseteq \mathbb{R}$.

6. Factorisation of the crossed product functor

Now we put our results together to factorise the crossed product functor $\mathfrak{Corr}(\mathcal{C}) \to \mathfrak{Corr}$ for a crossed module $\mathcal{C} = (G, H, \partial, c)$ of locally compact groups into "elementary" constructions. First we give more details on the strict extensions that decompose $\mathcal{C}$ into simpler building blocks.

The image $\partial(H)$ is a normal subgroup in G because $g\partial(h)g^{-1} = \partial_{c_g(h)}$. Hence $\overline{\partial(H)}$ is a closed normal subgroup in G and

$$\bar{\pi}_1(\mathcal{C}) := G/\overline{\partial(H)}$$

is a locally compact group. The closed subgroup

$$\pi_2(\mathcal{C}) := \ker \partial \subseteq H$$

is Abelian because $hkh^{-1} = c_{\partial(h)}(k)$ for all $h, k \in H$.

Since $\pi_2(\mathcal{C})$ is an Abelian locally compact group, there is a crossed module $\mathcal{C}_1$ with $H_1 = \pi_2(\mathcal{C})$ and trivial G_1 (and hence trivial ∂_1 and c_1). The G-action c on H leaves $\pi_2(\mathcal{C})$ invariant and hence descends to an action c_2 of G on $H_2 := H/\pi_2(\mathcal{C})$. Of course, ∂ descends to a map $\partial_2\colon H_2 \to G_2 = G$. This defines a crossed module of locally compact groups $\mathcal{C}_2$. The canonical maps $\mathcal{C}_1 \to \mathcal{C} \to \mathcal{C}_2$ are homomorphisms of crossed modules, and they clearly form a strict extension of crossed modules, based on the extensions of locally compact groups $\pi_2(\mathcal{C}) \rightarrowtail H \twoheadrightarrow H/\pi_2(\mathcal{C})$ and $0 \rightarrowtail G = G$.

There is a crossed module $\mathcal{C}_3$ with $G_3 := \overline{\partial(H)}$, $H_3 = H_2 = H/\pi_2(\mathcal{C})$, and $\partial_3\colon H_3 \to G_3$ and $c_3\colon G_3 \to \operatorname{Aut}(H_3)$ induced by ∂ and c. Let $\mathcal{C}_4$ be the crossed module with $G_4 = \bar{\pi}_1(\mathcal{C})$ and trivial H_4, ∂_4 and c_4. The obvious maps give homomorphisms of crossed modules $\mathcal{C}_3 \to \mathcal{C}_2 \to \mathcal{C}_4$; these form a strict extension of crossed modules because we have extensions $H_3 = H_2 \twoheadrightarrow 0$ and $G_3 \rightarrowtail G_2 \twoheadrightarrow G_4$ of locally compact groups.

The strict extensions above show that the crossed product functor for $\mathcal{C}$-actions factorises into the three partial crossed product functors with $\mathcal{C}_1$, $\mathcal{C}_3$ and $\mathcal{C}_4$. Now we analyse actions and crossed products for $\mathcal{C}_1$, $\mathcal{C}_4$, and $\mathcal{C}_3$, respectively.

Since a crossed module $\mathcal{C}_1$ of the form $(0, H_1, 0, 0)$ is Abelian, Theorem 4.5 shows that $\mathfrak{Corr}(\mathcal{C}_1)$ is equivalent to the 2-category of $\mathrm{C}_0(\widehat{H_1})$-C*-algebras, such that the crossed product by $\mathcal{C}_1$ corresponds to the functor that maps a $\mathrm{C}_0(\widehat{H_1})$-C*-algebra to its fibre at $1 \in \widehat{H_1}$. The case at hand is much easier than the general case of Theorem 4.5 because G is trivial. We simply observe that a $\mathcal{C}_1$-action on A is exactly the same as a nondegenerate *-homomorphism from $\mathrm{C}_0(\hat{H}) \cong \mathrm{C}^*(H)$ to the central multiplier algebra of A.

For crossed modules of the form $\mathcal{C}_4 = (G_4, 0, 0, 0)$, there is nothing to analyse: actions of this crossed module are the same as actions of the locally compact group G_4, and the crossed product functor is also the same as for group actions. We already showed in [2] that group actions by correspondences are equivalent to saturated Fell bundles. The crossed product is the cross-sectional C*-algebra of a Fell bundle. By the Packer–Raeburn Stabilisation Trick (see also Theorem [2, Theorem 5.3]), we may replace G_4-actions by correspondences by ordinary continuous group actions on a stabilisation. This replaces the crossed product functor for actions of $\mathcal{C}_4$ by a classical crossed product construction for actions of the locally compact group G_4.

Now we study crossed products by the thin crossed module

$$\mathcal{C}_3 = (\overline{\partial H}, H/\ker\partial, \partial_3, c_3),$$

using the results in Section 3.1 to replace $\mathcal{C}_3$ by an equivalent Abelian crossed module $\mathcal{C}_5$. By Theorem 3.11, such a Abelian model for $\mathcal{C}_3$ exists if both G and H are Lie groups. More generally, Lemma 3.15 gives a Abelian model if $H/\ker\partial$ has a compactly generated subgroup A for which $\overline{\partial(A)}$ is open in $\overline{\partial(H)}$; in particular, this happens if H itself is compactly generated. Theorem 3.14 also gives a necessary and sufficient condition for a Abelian model to exist; but this criterion does not explain why this happens so often.

Assume that $\mathcal{C}_3$ is equivalent to an Abelian crossed module $\mathcal{C}_5 = (G_5, H_5, \partial_5, c_5)$; even better, we can achieve that G_5 is compact Abelian, H_5 is discrete, and c_5 is trivial. Since $\mathcal{C}_3$ is thin, so is $\mathcal{C}_5$, that is, ∂_5 is an injective map with dense range. By Theorem 3.5, the 2-categories $\mathfrak{Corr}(\mathcal{C}_3)$ and $\mathfrak{Corr}(\mathcal{C}_5)$ of actions of $\mathcal{C}_3$ and $\mathcal{C}_5$ by correspondences are equivalent, in such a way that the crossed product functors on both categories are identified. Moreover, the proof shows immediately that the underlying C*-algebra is not changed: an action of $\mathcal{C}_3$ becomes a $\mathcal{C}_5$-action on the same C*-algebra. For crossed modules of Lie groups, we have explained in Section 3 how to construct $\mathcal{C}_5$ explicitly out of $\mathcal{C}_3$.

Theorem 4.5 shows that $\mathfrak{Corr}(\mathcal{C}_5)$ is equivalent to the 2-category of actions of the groupoid $\widehat{G_5} \ltimes \widehat{H_5}$; this equivalence maps a $\mathcal{C}_5$-action to the crossed product by G_5 equipped with a canonical $\mathrm{C}_0(\widehat{H_5})$-C*-algebra structure and the dual action of $\widehat{G_5}$; the crossed product by $\mathcal{C}_5$ corresponds to taking the fibre at 1 for the $\mathrm{C}_0(\widehat{H_5})$-C*-algebra structure. Thus after an equivalence

$$\mathfrak{Corr}(\mathcal{C}_3) \simeq \mathfrak{Corr}(\mathcal{C}_5) \simeq \mathfrak{Corr}(\widehat{G_5} \ltimes \widehat{H_5})$$

that on the underlying C*-algebras takes a crossed product with the Abelian compact group G_5, the crossed product with $\mathcal{C}_3$ becomes a fibre restriction functor.

Each of the steps above is either taking a crossed product by a locally compact group or a fibre in a $\mathrm{C}_0(X)$-C*-algebra. Thus the crossed product factorises into these more classical constructions.

6.1. **Computing K-theory of crossed module crossed products.** In the localisation formulation of [11], the Baum–Connes assembly map for a locally compact group G compares the K-theory of the reduced crossed product with a more topological invariant that uses only crossed products for restrictions of the action to compact subgroups of G. Its assertion is therefore trivial if G is itself compact. Crossed products for compact groups are an "elementary" operation for K-theory purposes in the sense that there is no better way to compute the K-theory than the direct one. Crossed products for non-compact groups are not "elementary" in this sense because the Baum–Connes conjecture (if true) allows us to reduce the K-theory computation to K-theory computations for compact subgroups and some algebraic topology to assemble the results of these computations.

Taking the fibre in a $\mathrm{C}_0(X)$-C*-algebra seems to be an operation that is also "elementary" in the above sense. At least, we know of no better way to compute the K-theory of a fibre than the direct one. Notice that $\mathrm{C}_0(X)$-C*-algebras need not be locally trivial.

The first three steps in our decomposition of the crossed product for a general crossed module are taking a fibre in a $\mathrm{C}_0(X)$-C*-algebra, taking a crossed product for a compact group, and taking again a fibre in a $\mathrm{C}_0(X)$-C*-algebra. These three steps thus seem "elementary" for K-theory purposes. The remaining fourth step is the (full) crossed product by the locally compact group $G/\overline{\partial(H)}$. Many results are available about the K-theory of such crossed products.

Hence our decomposition of crossed module crossed products also gives us a useful recipe for computing their K-theory. This recipe is, however, quite different from the localisation approach for groups in [11].

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